APPLIED MECHANICS

ENGINEERING MECHANICS

CHAPTER 1 :Introduction And Review of Mathematics

1.1 Introduction

Basic mechanics involves the study of two principal areas – statics and dynamics.

Statics & Dynamics.exe Bridges.exe

Statics is the study of forces on objects or bodies which are at rest or moving at a constant velocity, and the forces are in balance, or in *static equilibrium*.

equilibrium.exe

A ball at rest may have several forces acting on it, such as gravitational force (weight) and a force opposing that gravity (reaction). The ball is at rest or static, has forces in balance or **EQUILIBRIUM**

Dynamics is the study of forces on moving bodies, and the forces are in *dynamic equilibrium*.

The concept of applied mechanics is useful when it comes to analyzing stress, designing of machine structures and hydraulics, etc.

Beam under stress.exe Hydranlics.exe

There are only seven basic units in the SI system but only three are frequently used in statics and dynamics:

	Physical Quantity	Unit	Symbol
1.	Length	meter	m
2.	Mass	kilogram	kg
3.	Time	second	s

For large or small figures, multiples or submultiples are used. For example:

Multiples

1 kilogram is 1 kg or 10^3 g.

1 megagram is 1 Mg or 10^6 g.

1 gigagram is 1 Gg or 10^9g.

Submultiples

1 millimeter is 1 mm or 10^{-3} m.

1 micrometer is 1 μm or 10^{-6}m.

1 nanometer is 1 nm or 10^{-9}m.

The following SI derived units are frequently used in this course:

Force – The unit of force is the newton (N) .
1 newton is the force applied to a 1 kg mass to give it an acceleration of 1 m/s^2 (i.e 1 N = 1 kg.m/s^2).

Forces.exe

Or : Force = mass x acceleration
= kg x m/s^2 = kg m/s^2

Hence a 1 kg mass has a force or weight due to gravity, equal to (1 kg x 9.81 m/s$^{2)}$ = 9.81 N,
where g = 9.81 m/s^2 .

Moment – It is the product of a force and its perpendicular distance, and the unit is newton-meter or N-m.

1.2 Mathematics Required

The followings are the mathematics skills that are important for this module :

- **Quadratic equations**
- **Simultaneous equations**
- **Trigonometry functions of a right-angle triangle**
- **Sine and cosine rules**

1.2.1 Quadratic Equations

The equation has the standard form as follows

$$ax^2 + bx + c = 0 \qquad (1.1)$$

The standard solution to this equation is

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \qquad (1.2)$$

Example 1

Solve for x in the equation $5x^2 + 12x - 2 = 0$. Comparing equation 1.1 above, and substitute a=5, b=12 and c= –2 into equation 1.2, the solution is :

$$X = \frac{-12 \pm \sqrt{[(12)^2 - 4(5)(-2)]}}{2(5)}$$

$$= \frac{-12 \pm 13.56}{10}$$

$$= +0.156 \text{ or } -2.56$$

1.2.2 Simultaneous Equation

The equation has two unknowns x and y in the form of

$$ax + by + c = 0 \qquad (1.3)$$

$$px + qy + r = 0 \qquad (1.4)$$

<u>Example 2</u>

Find the values of x and y satisfying the given equations:

$$4x + 3y + 10 = 0 \qquad (1)$$

$$20x + 30y + 5 = 0 \qquad (2)$$

There are 2 methods of solving these equations

Method of Substitution

We can start by expressing x in terms of y, or y in terms of x.
Let us choose to express x in terms of y, thus from (1)

$$x = \frac{-3y - 10}{4} \qquad (3)$$

Substituting (3) into (2) , yielding

$$20\frac{(-3y - 10)}{4} + 30y + 5 = 0$$

$$-15y - 50 + 30y + 5 = 0$$

$$15y - 45 = 0$$

$$y = \frac{45}{15} = 3$$

To find x, substitute the value of y into (3)

$$x = \frac{(-3 \times 3 - 10)}{4} = \frac{-19}{4}$$

Method of Elimination

This method looks for a way to eliminate one of the unknowns.

This can be done by making the constant factor of that unknown or variable the same in both equations by multiplying or dividing one equation by a selected constant:

$4x + 3y + 10 = 0$ (1)

$20x + 30y + 5 = 0$ (2)

Divide (2) by 5

$4x + 6y + 1 = 0$ (3)

Subtract (3) by (1)

$3y - 9 = 0$

$y = 3$

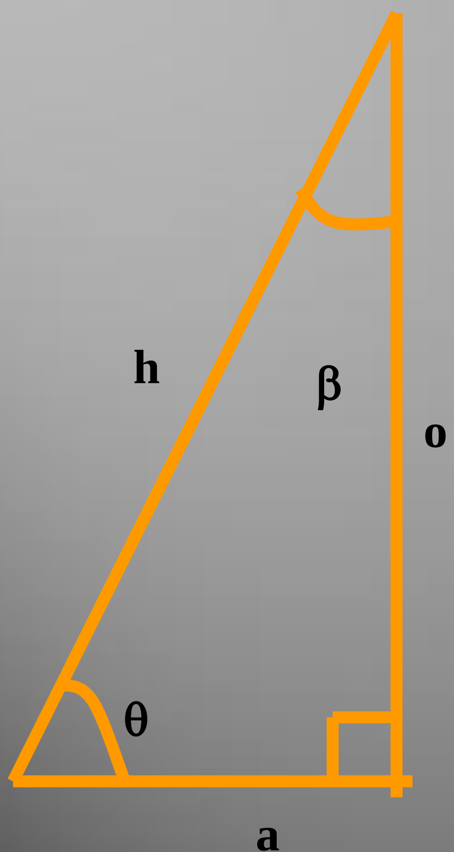
Substitute the value of y into (1) or (2)

$4x + 3(3) + 10 = 0$

$4x = -9 - 10$

$x = -\frac{19}{4}$

1.2.3 Trigonometry Functions Of A Right-Angle Triangle



$$\text{sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{o}{h} = \text{cosine } \beta \quad (1.5)$$

$$\text{cosine } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{a}{h} = \text{sine } \beta \quad (1.6)$$

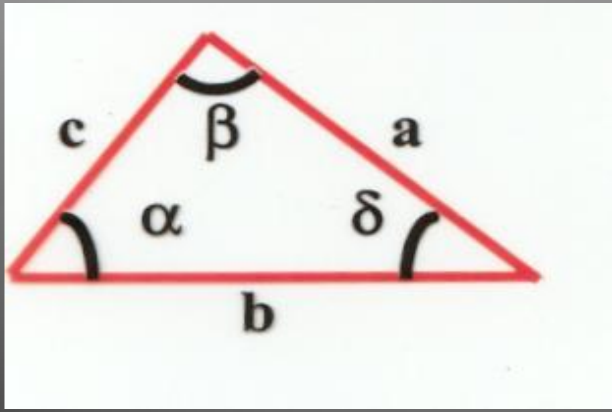
$$\text{tangent } \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{o}{a} \quad (1.7)$$

$$\text{tangent } \theta = \frac{\sin \theta}{\cos\theta}$$

For triangles that are not right-angle, the following two laws are important in vector algebra introduced in chapter two later:

Cosine Rule

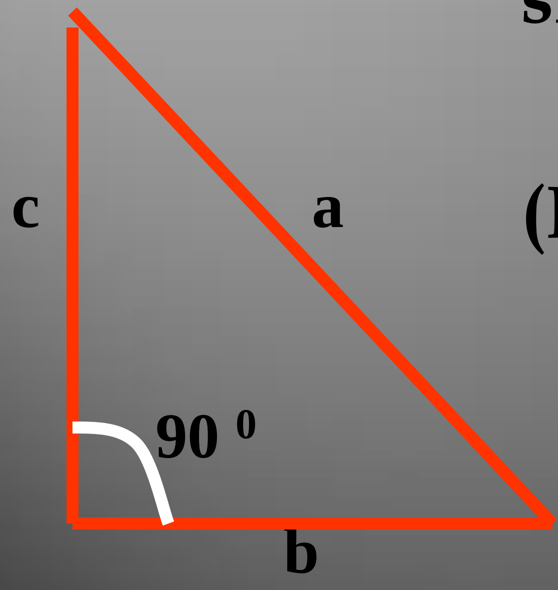
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad (1.8)$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\delta$$



Sine Rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\delta}$$

If the cosine rule is applied to a right-angle triangle where $\alpha = 90^0$, and applying to equation (1.8),



i.e. $a^2 = b^2 + c^2 - 2bc \cos 90^0$

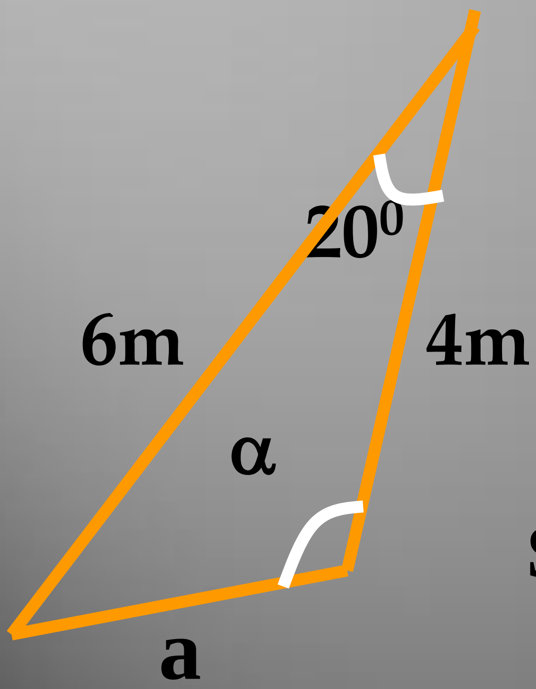
since $\cos 90^0 = 0$

$$a^2 = b^2 + c^2$$

(Pythagoras Theorem)

Example 3

Find the length of the unknown side a and the angle α.



Cosine rule : $a^2 = b^2+c^2-2bc\cos\theta$

i.e. $a^2 = 6^2+4^2-2\text{x}6\text{x}4\cos 20^0$

$= 36 +16-6\text{x}4\text{x}\cos 20^0$

$= 6.9$

$a = 2.63\text{m}$

Sine rule : $\dfrac{2.63}{\sin 20^0} = \dfrac{6}{\sin \alpha}$

$\text{sine } \alpha = 6 \text{ x } \dfrac{\sin 20^0}{2.63}$

$\alpha = 51.3^0$

But we know this to be in the second quadrant,

Hence $\alpha = 180 - 51.4 = 128.6\,^{0}$

Check : $6^2 = 2.63^2 - 4^2 - 2x2.63x4 \cos\alpha$

$$\cos\alpha = \frac{2.63^2 + 4^2 - 6^2}{2x2.63x4}$$

$$= -0.634$$

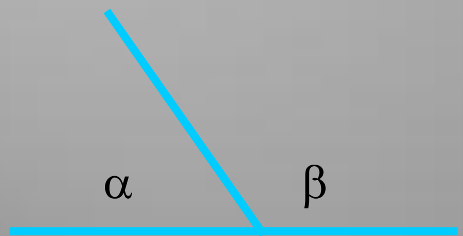
$$\alpha = 128.6\,^{0}$$

1.2.5 Geometry

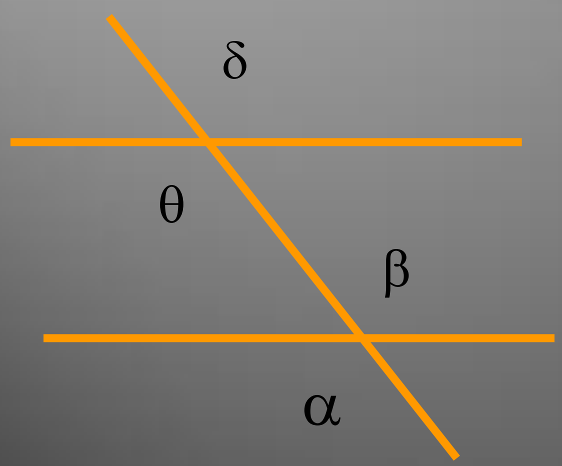
Some of the basic rules are shown below:

Sum of supplementary angles = 180 ^{0}

$$\alpha + \beta = 180^0$$



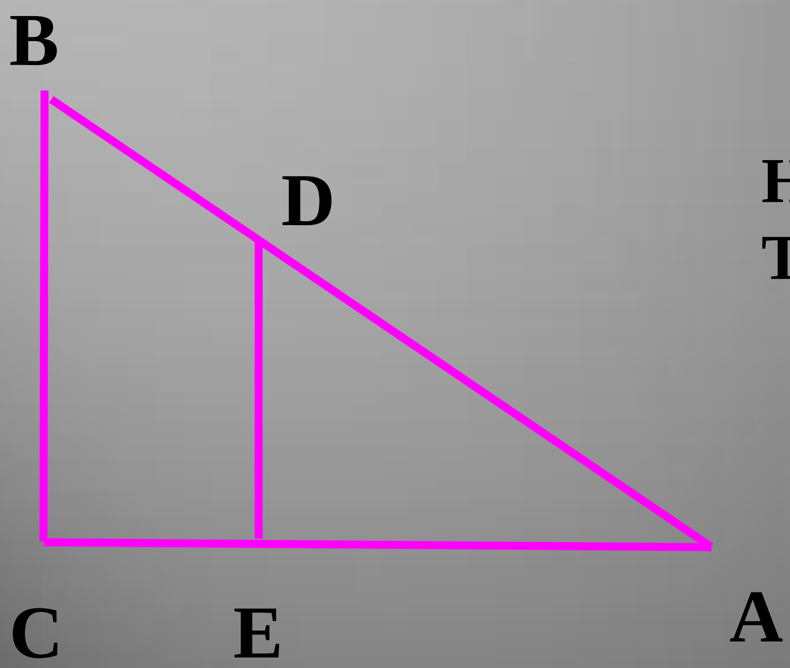
A straight line intersecting two parallel lines



$$\delta = \beta, \quad \theta = \alpha$$

$$\delta = \theta, \quad \beta = \alpha$$

Similar triangles ABC and ADE, by proportion



$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

Hence if AB = 6, AD = 3 and BC = 4,
Then,

$$\frac{6}{3} = \frac{4}{DE}$$

$$DE = \frac{(3 \times 4)}{3}$$
$$= 2$$

End of Chapter 1

FORCE SYSTEMS

2-D Force Systems

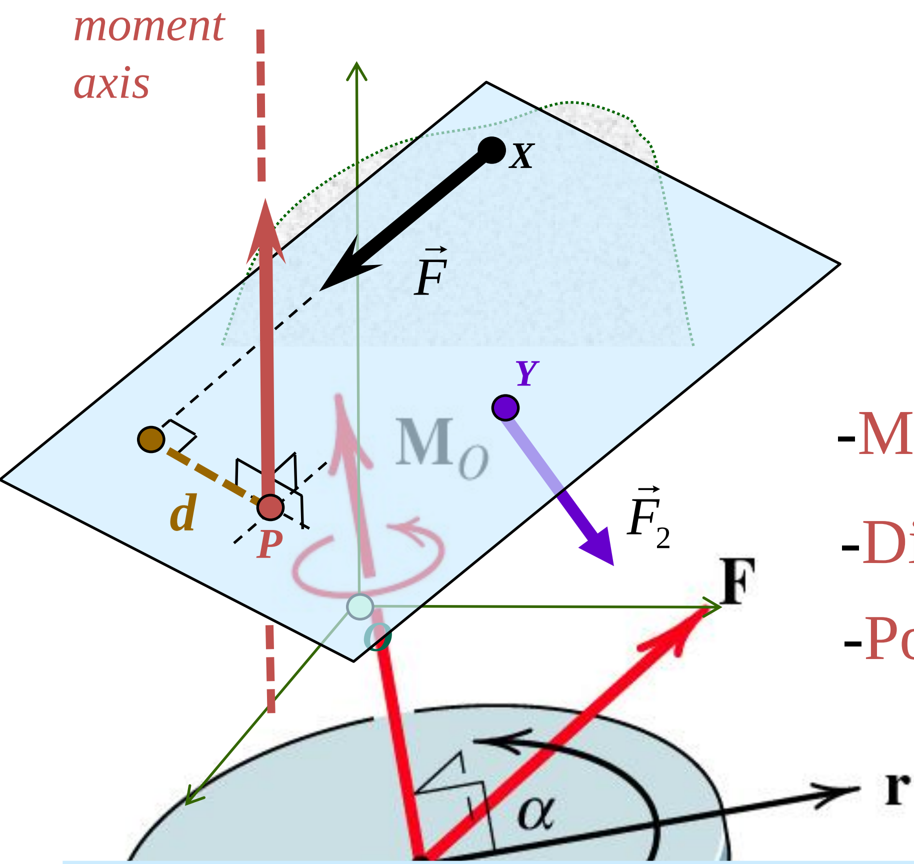
- Force
- Moment,Couple
- Resultants

3-D Force Systems

- Force
- Moment, Couple
- Resultants

3D-Force Systems

Rectangular Components, Moment, Couple, Resultants



Moment (3D)

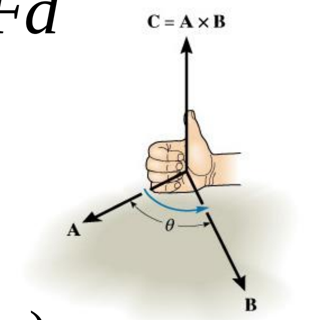
Moment about <u>point P</u> :

$$\vec{M}_{P,\vec{F}} = \vec{r} \times \vec{F}$$

-Magnitude: $|\vec{r}\,\|\,\vec{F}\,|\sin\alpha = Fd$

-Direction: right-hand rule

-Point of application: point O



(Unit: newton-meters, N-m)

In 3D, forces (generally) are not in the same plane.

In many cases on 3D, *d* (the perpendicular distance) is hard to find. It is usually easier to find the moment by using the vector approach with cross product multiplication.

Cross Product

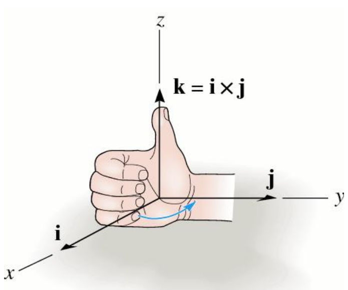
$$\vec{M}_{o,\vec{F}} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

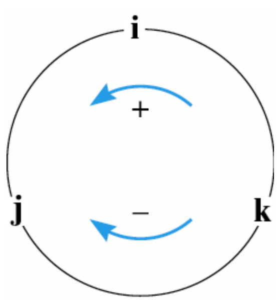
For element **i**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z - A_zB_y)$

For element **j**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_xB_z - A_zB_x)$

For element **k**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_xB_y - A_yB_x)$

$$\vec{M}_o = (r_x\hat{i} + r_y\hat{j} + r_z\hat{k}) \times (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})$$





$$= \cancel{r_xF_x(\hat{i}\times\hat{i})} + r_xF_y(\hat{i}\times\hat{j}) + r_xF_z(\hat{i}\times\hat{k}) +$$

$$ {}_yF_x(\hat{j}\times\hat{i}) + \cancel{r_yF_y(\hat{j}\times\hat{j})} + r_yF_z(\hat{j}\times\hat{k}) +$$

$$ {}_zF_x(\hat{k}\times\hat{i}) + r_zF_y(\hat{k}\times\hat{j}) + \cancel{r_zF_z(\hat{k}\times\hat{k})}$$

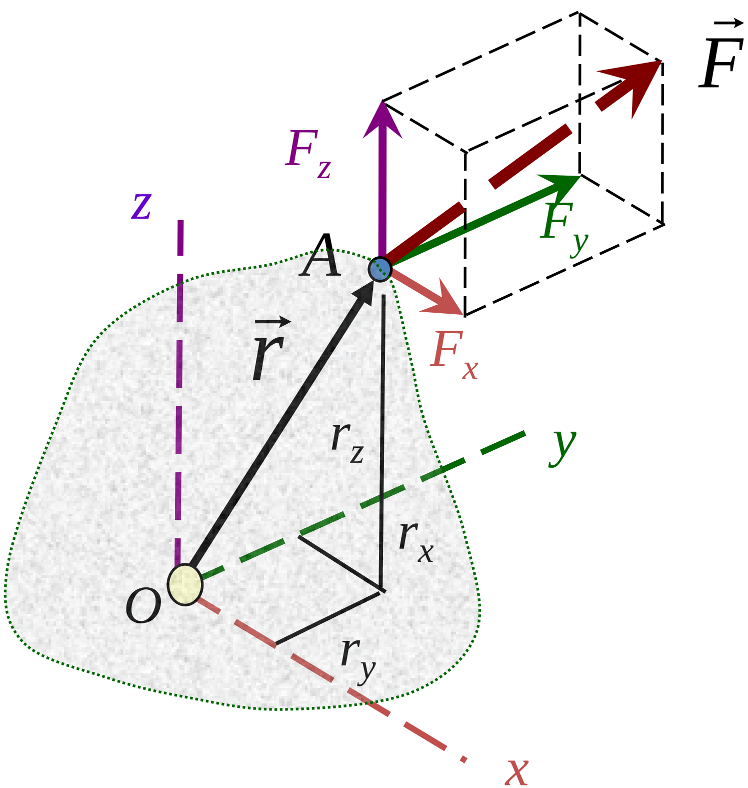
Beware: xyz axis must complies with right-hand rule

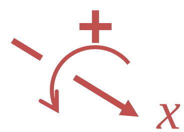
$$\vec{M}_o = (r_yF_z - r_zF_y)\hat{i} + (r_zF_x - r_xF_z)\hat{j} + (r_xF_y - r_yF_x)\hat{k}$$

Moment (Cross Product)

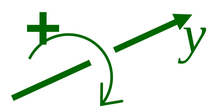
$$\vec{M}_o = (r_y F_z - r_z F_y)\,\hat{i} + (r_z F_x - r_x F_z)\,\hat{j} + (r_x F_y - r_y F_x)\,\hat{k}$$

Physical Meaning

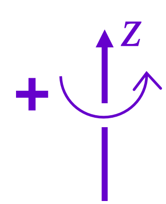




$$M_x = -\,F_y r_z + F_z r_y$$

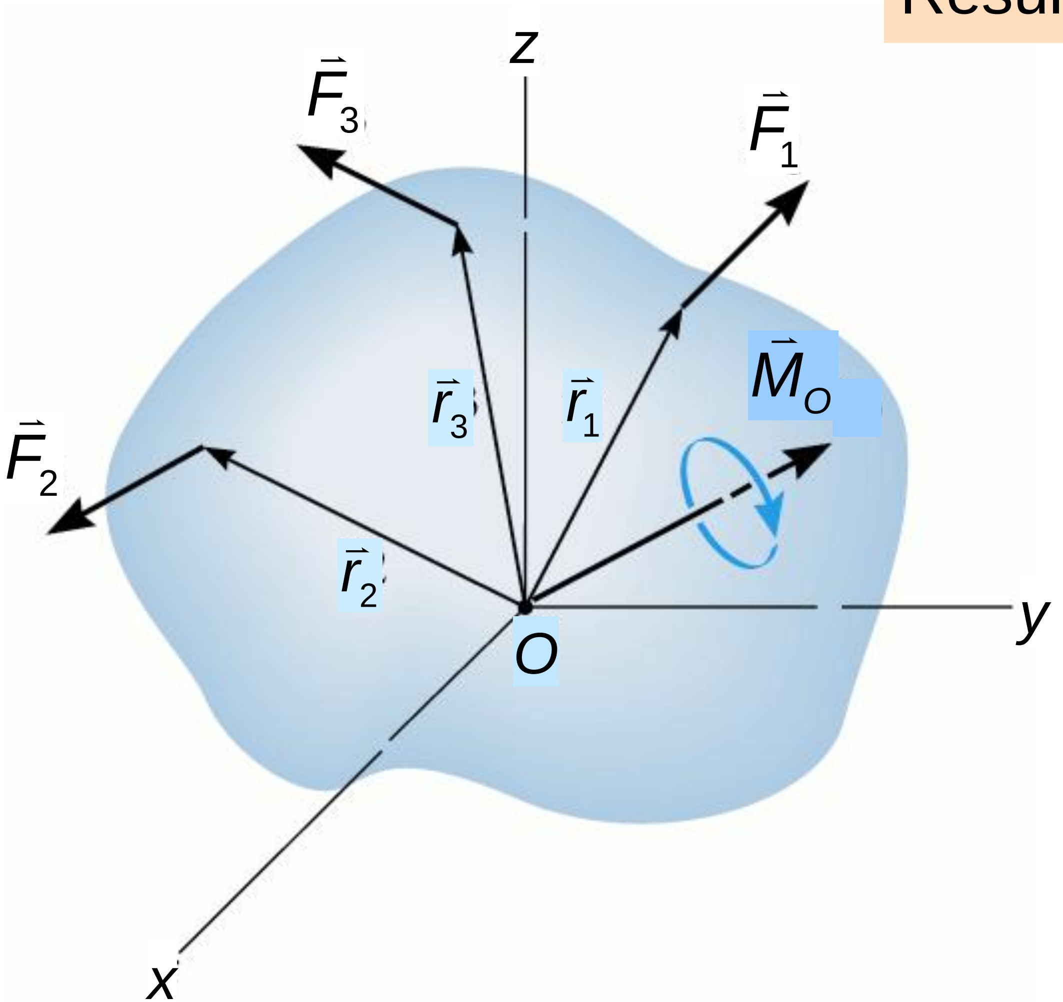


$$M_y = +F_x r_z - F_z r_x$$



$$M_z = -F_x r_y + F_y r_x$$

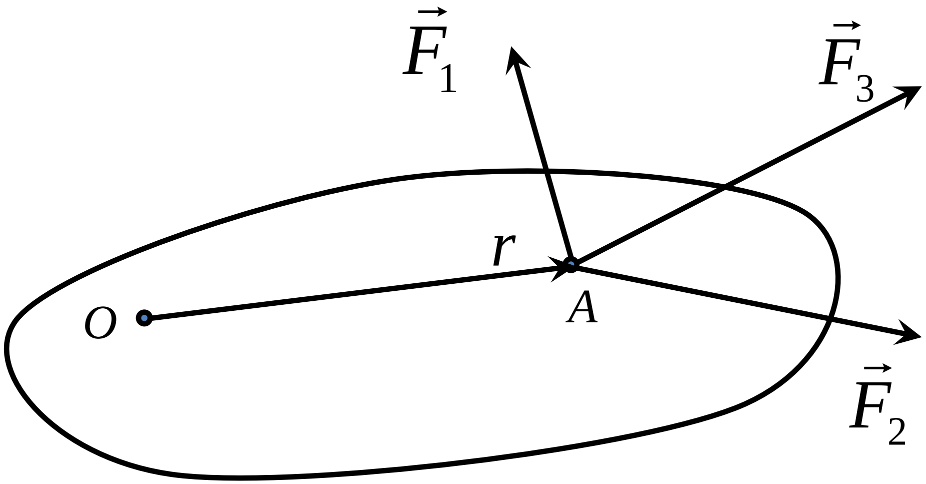
Moment About a Point #4



Resultant Moment of Forces

$$\vec{M}_O = \sum_i (\vec{r}_i \times \vec{F}_i)$$

Varignon's Theorem (Principal of Moment)



- Two or more concurrent forces

- their moments about a point may be found in two ways

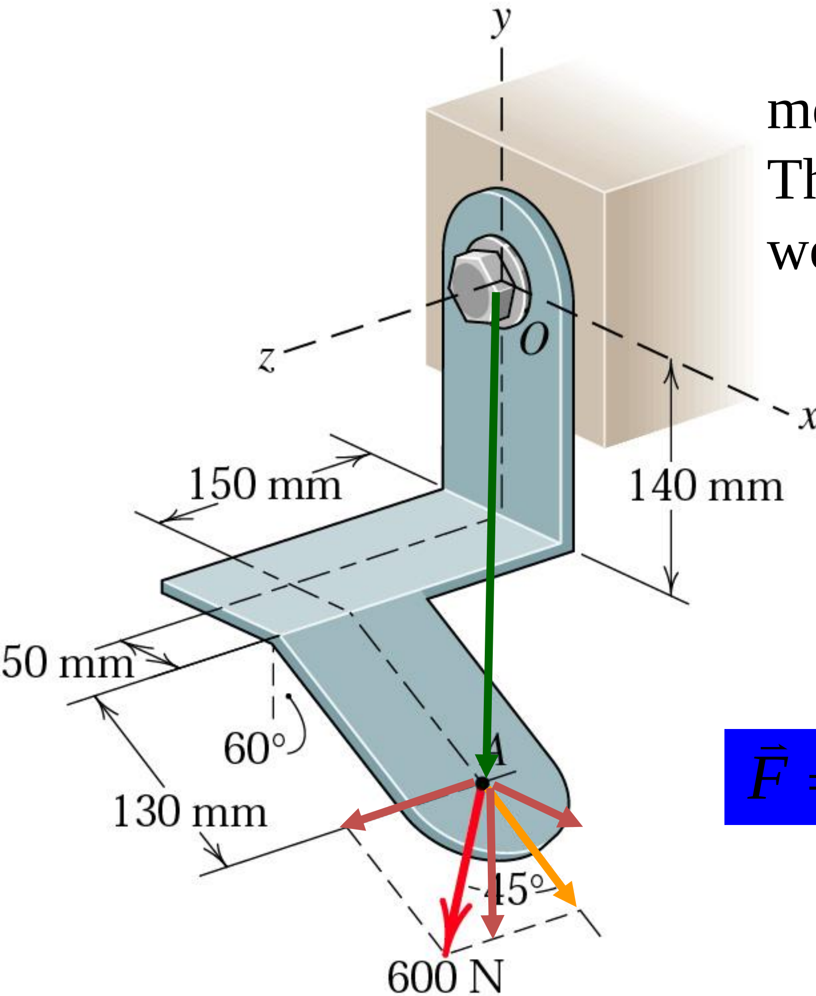
- for nonconcurrent forces see Resultants sections (2D - 2/6, 3D- 2/9)

$$\vec{M}_o = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \ldots = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots)$$
$$= \vec{r} \times (\sum \vec{F})$$

$$\boxed{\vec{M}_o = \sum (\vec{r} \times \vec{F}) = \vec{r} \times \vec{R}}$$

- Sum of the moments of a system of concurrent forces about a given point equals the moment of their sum about the same point

Determine the vector expression for the moment $\vec{M}_o$ of the 600-N force about point **O**. The design specification for the bolt at **O** would require this result.



$$\vec{M}_O = \vec{r}_{OA} \times \vec{F}$$

$$\vec{r}_{OA} = (0.05 + 0.13\sin 60^\circ)\,\hat{i} - (0.14 + 0.13\cos 60^\circ)\,\hat{j} + 0.15\,\hat{k}$$

$$\vec{r}_{OA} = 0.1626\,\hat{i} - 0.205\,\hat{j} + 0.15\,\hat{k}$$

$$\vec{F} = ?$$

$$F_z = 600\sin 45^o \qquad F_{xy} = 600\cos 45^o$$

$$F_x = F_{xy}\sin 60^0 = 600\cos 45^o \sin 60^o$$

$$F_y = -F_{xy}\cos 60^0 = -600\cos 45^o \cos 60^o$$

$$\vec{F} = 600(0.612\,\hat{i} - 0.354\,\hat{j} + 0.707\,\hat{k})$$

$$= -55.2\,\hat{i} - 13.86\,\hat{j} + 40.8\,\hat{k}$$

$$\vec{M}_{O,\vec{F}} = \vec{r}_{OA} \times \vec{F}$$

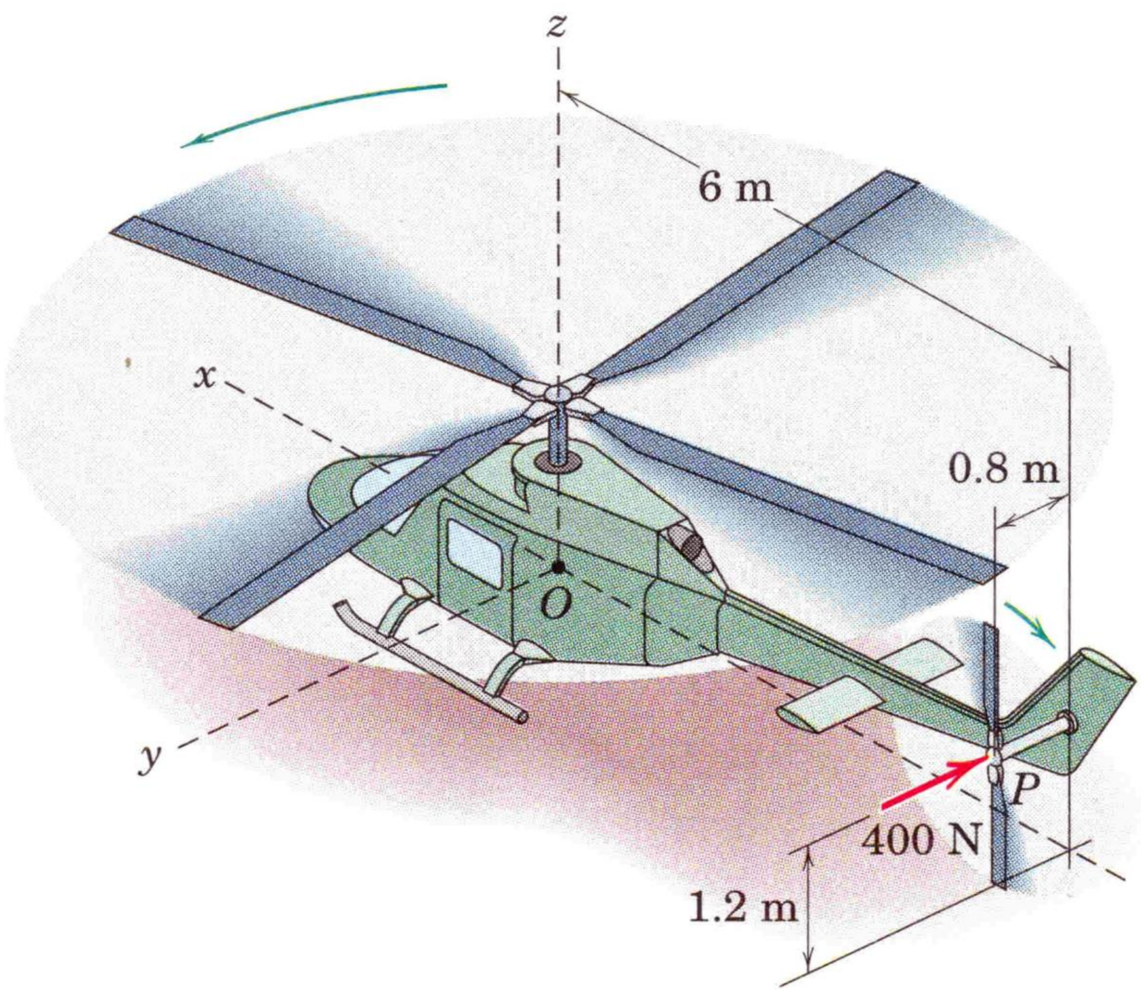
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.1626 & -0.205 & 0.15 \\ 367 & -212 & 424 \end{vmatrix}$$

$$= 367\,\hat{i} - 212\,\hat{j} + 424\,\hat{k} \quad \text{N}$$

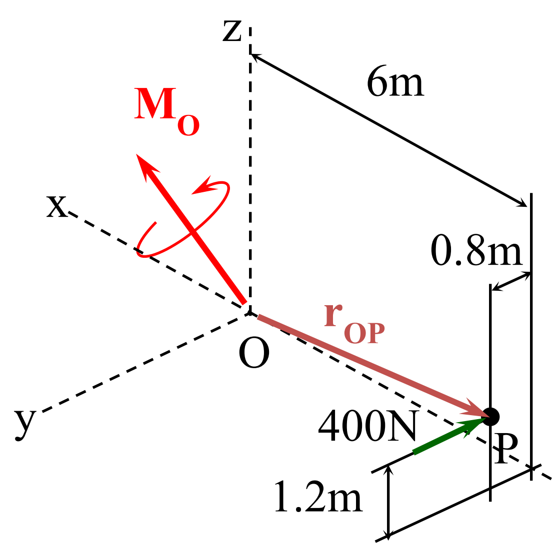
Ans

2/114 The helicopter of Prob. 2/58 is redrawn here with certain three-dimensional geometry given. During a ground test, a 400-N aerodynamic force is applied to the tail rotor at P as shown. Determine the moment of this force about point O of the airframe.

Using: $\vec{r}_{OP}$ (3D Vector)



Problem 2/114



$$\vec{r}_{OP} = -6\hat{i} + 0.8\hat{j} + 1.2\hat{k}$$

$$\vec{F} = -400\hat{j}$$

$$\vec{M}_O = \vec{r}_{OP} \times \vec{F}$$

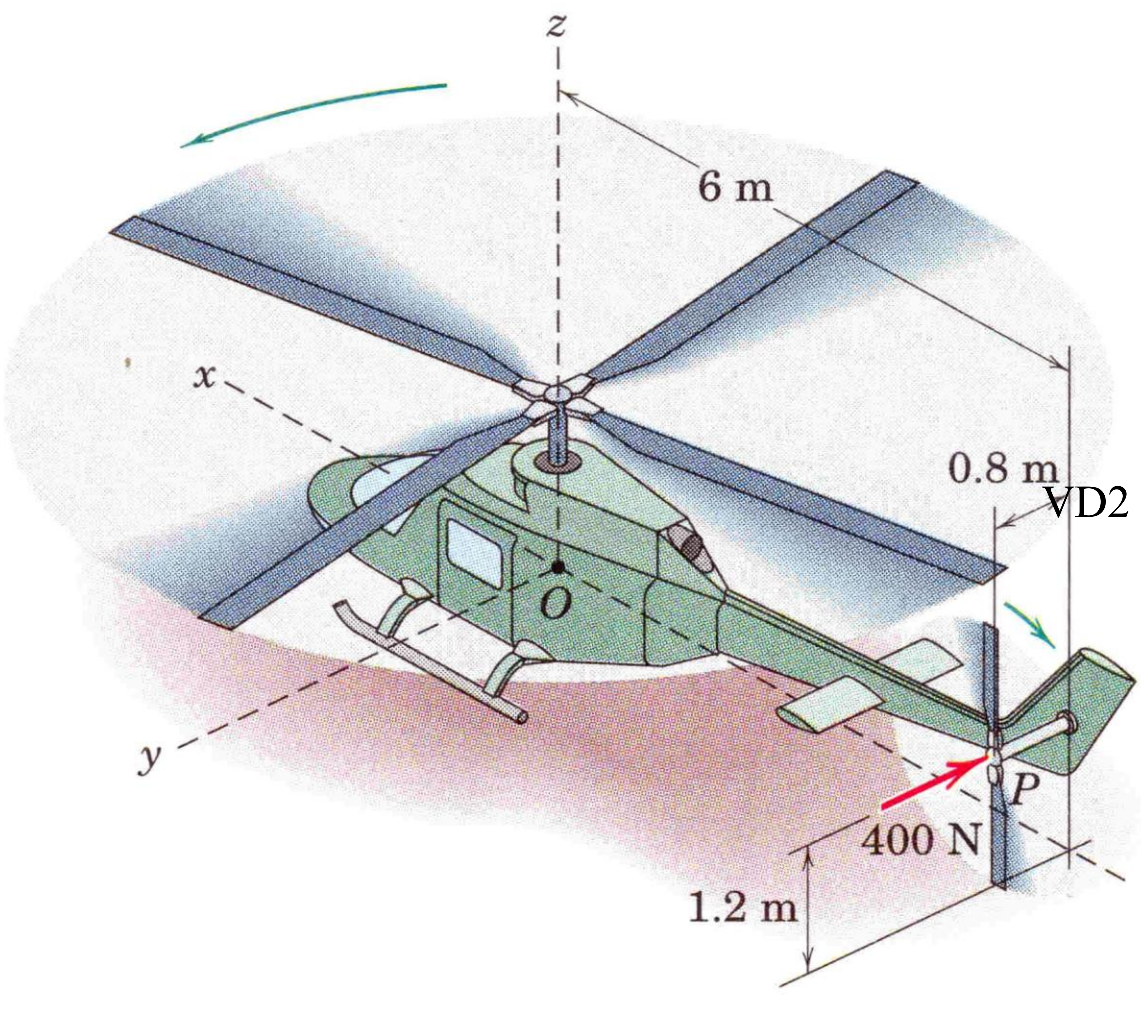
$$= \begin{vmatrix} i & \hat{j} & \hat{k} \\ -6 & 0.8 & 1.2 \\ 0 & -400 & 0 \end{vmatrix} = 480\hat{i} + 2400\hat{k}$$

N-m

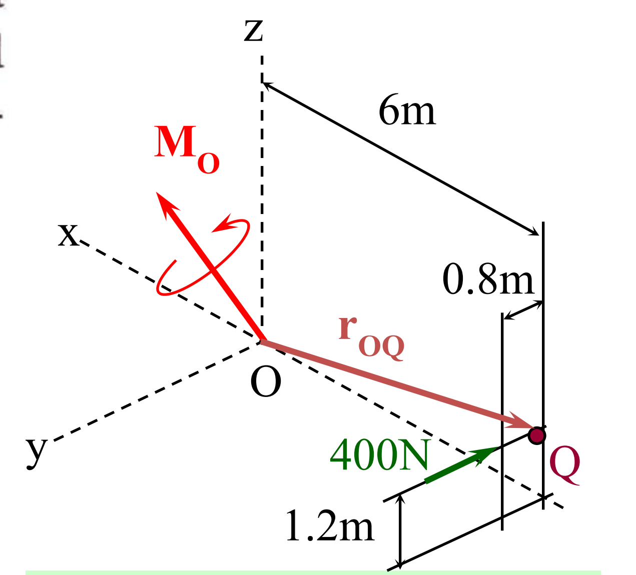
Ans

2/114 The helicopter of Prob. 2/58 is redrawn here with certain three-dimensional geometry given. During a ground test, a 400-N aerodynamic force is applied to the tail rotor at *P* as shown. Determine the moment of this force about point *O* of the airframe.

Using: $\vec{r}_{OQ}$ (3D vector)



Problem 2/114



$$\vec{r}_{OQ} = -6\hat{i} + 0\hat{j} + 1.2\hat{k}$$

$$\vec{F} = -400\hat{j}$$

$$\vec{M}_O = \vec{r}_{OQ} \times \vec{F}$$

$$= \begin{vmatrix} i & \hat{j} & \hat{k} \\ -6 & 0 & 1.2 \\ 0 & -400 & 0 \end{vmatrix} = 480\hat{i} + 2400\hat{k}$$

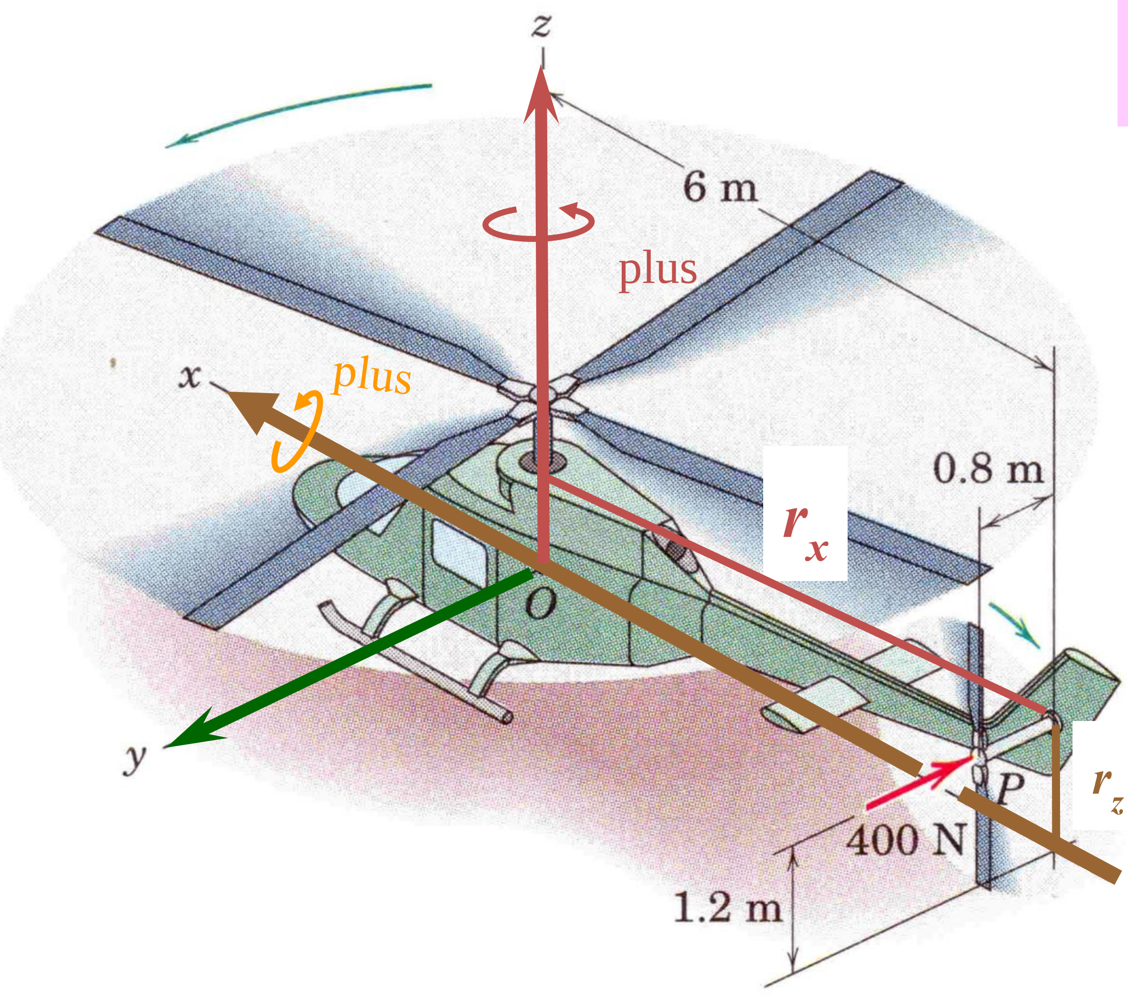
N-m

<u>***Ans***</u>

2/114 The helicopter of Prob. 2/58 is redrawn here with certain three-dimensional geometry given. During a ground test, a 400-N aerodynamic force is applied to the tail rotor at P as shown. Determine the moment of this force about point O of the airframe.

Using: $\vec{r}_{OQ}$ (Scalar × 3Plane)

$$\vec{M}_O = \boxed{}\,\hat{i} + \boxed{}\,\hat{j} + \boxed{}\,\hat{k}$$



Problem 2/114

$$M_{O,x} = +|r_z||F_y| = 1.2 \times 400 = 480$$

$$M_{O,y} = 0$$

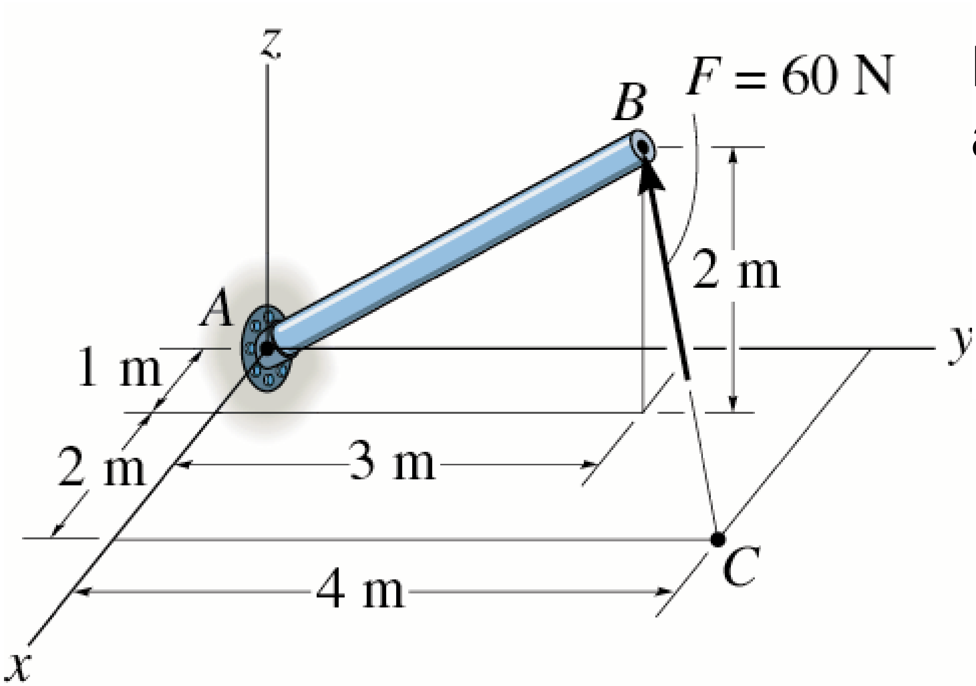
$$M_{O,z} = +|r_x||F_y| = 6 \times 400 = 2400$$

$$\vec{M}_O = 480\hat{i} + 2400\hat{k}$$ N-m

Ans

Not-Recommended Method

Example Hibbeler Ex 4-4 #1

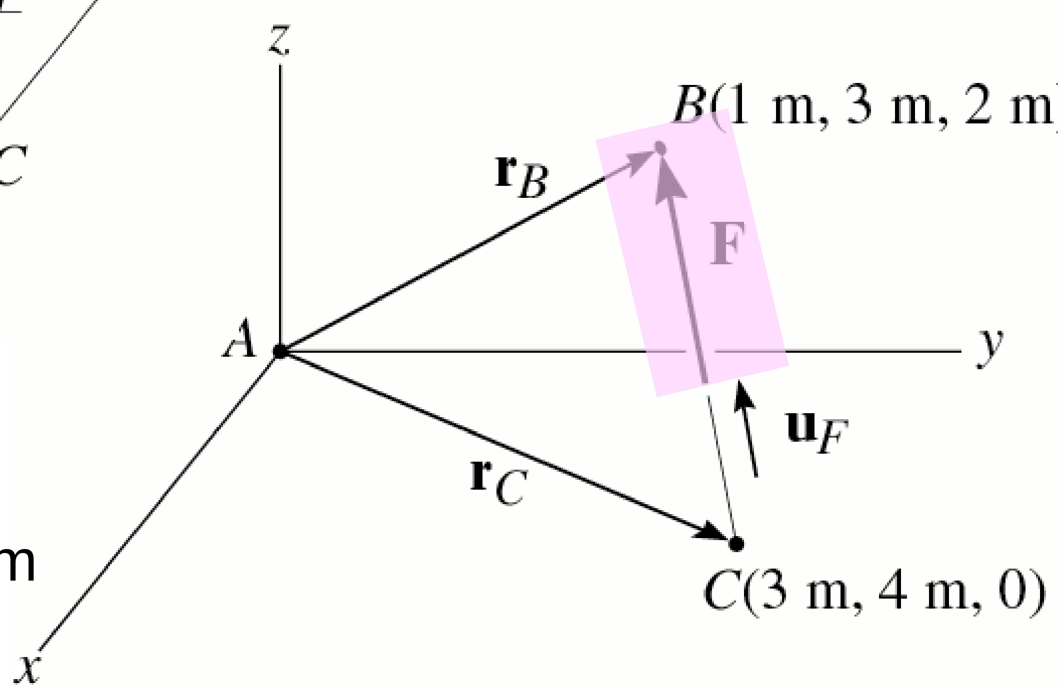


Determine the moment about the support at *A*.

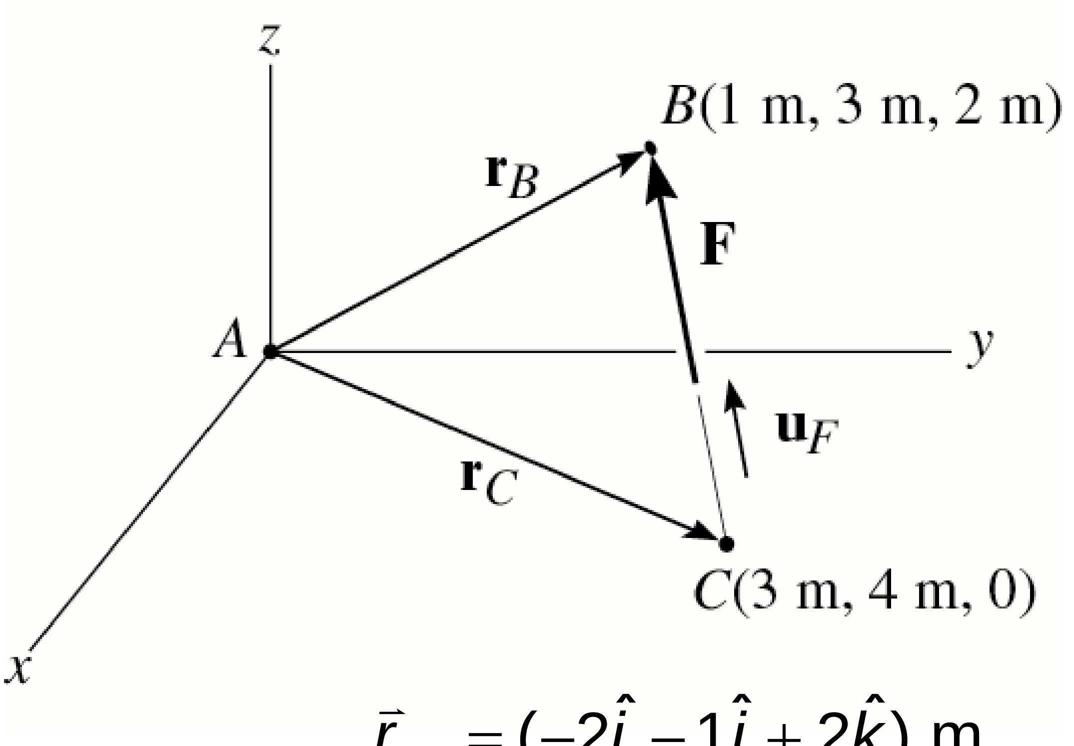
$$\vec{r}_B = (1\hat{i} + 3\hat{j} + 2\hat{k})\ \text{m}$$

$$\vec{r}_C = (3\hat{i} + 4\hat{j})\ \text{m}$$

$$\vec{r}_{CB} = \vec{r}_B - \vec{r}_C = (-2\hat{i} - 1\hat{j} + 2\hat{k})\ \text{m}$$



Example Hibbeler Ex 4-4 #2

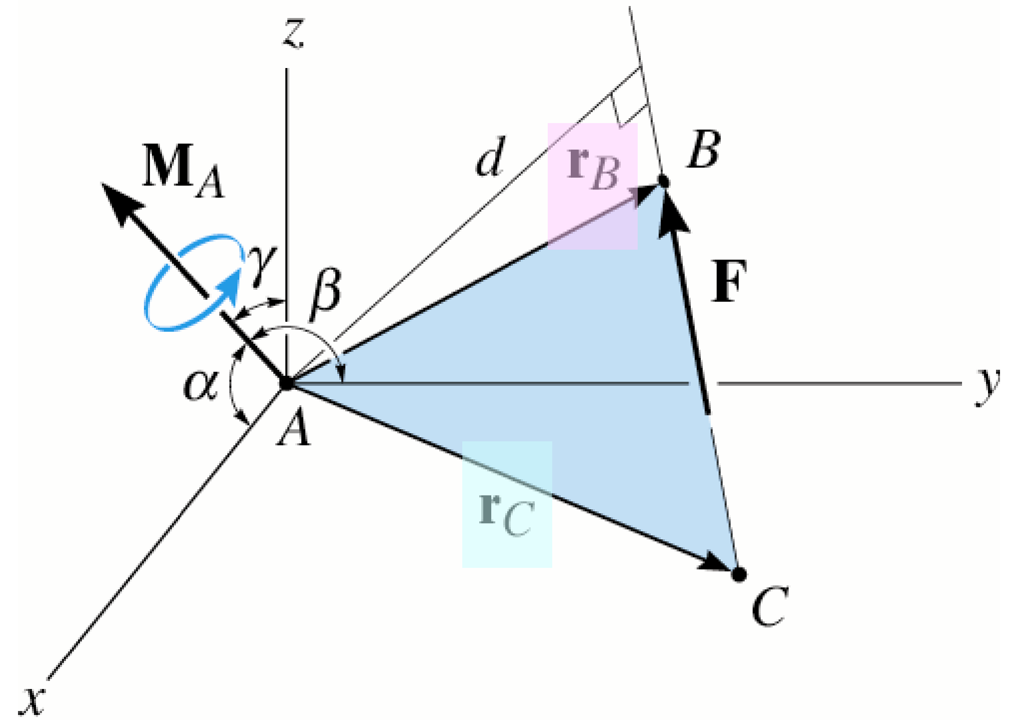


$$\vec{r}_{CB} = (-2\hat{i} - 1\hat{j} + 2\hat{k})\ \text{m}$$

$$\vec{u}_F = \frac{\vec{r}_{CB}}{r_{CB}} = \frac{-2\hat{i} - 1\hat{j} + 2\hat{k}}{\sqrt{(-2)^2 + (-1)^2 + 2^2}} = \frac{1}{3}(-2\hat{i} - 1\hat{j} + 2\hat{k})$$

$$\vec{F} = (60\ \text{N})\vec{u}_F = (-40\hat{i} - 20\hat{j} + 40\hat{k})\ \text{N}$$

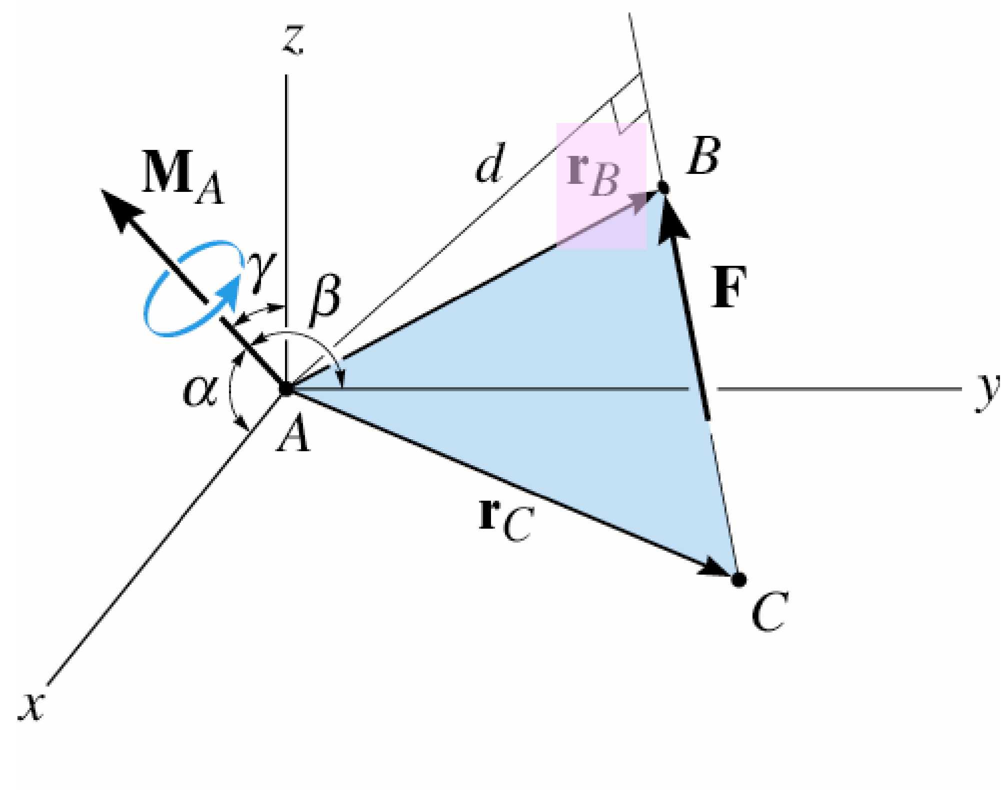
Example Hibbeler Ex 4-4 #3



$$\vec{M}_A = \vec{r}_C \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -40 & -20 & 40 \end{vmatrix} = (160\hat{i} - 120\hat{j} + 100\hat{k})\ \text{N}\cdot\text{m}$$

$$M_A = \sqrt{(160)^2 + (-120)^2 + (100)^2} = 223.61 = 224\ \text{N}\cdot\text{m} \qquad \#$$

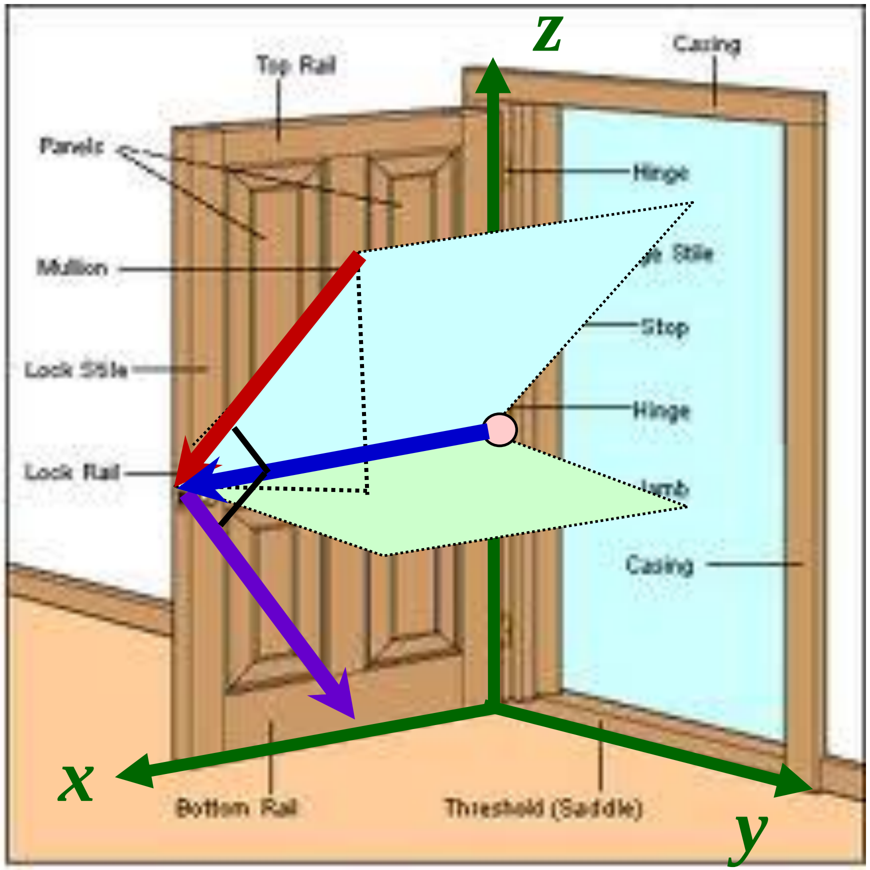
Example Hibbeler Ex 4-4 #4

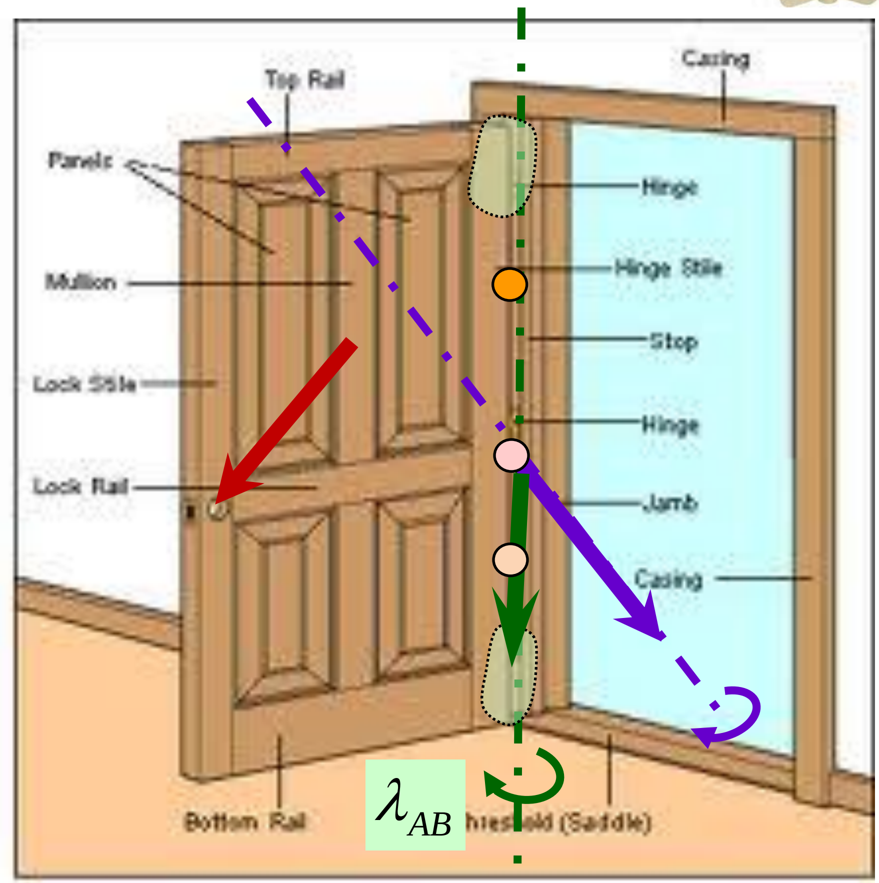


$$\vec{M}_A = \vec{r}_C \times \vec{F} = \vec{r}_B \times \vec{F}$$

$$\vec{M}_A = \vec{r}_B \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ -40 & -20 & 40 \end{vmatrix} \ \mathrm{N \cdot m}$$

$$\vec{M}_A = (160\hat{i} - 120\hat{j} + 100\hat{k}) \ \mathrm{N \cdot m}$$





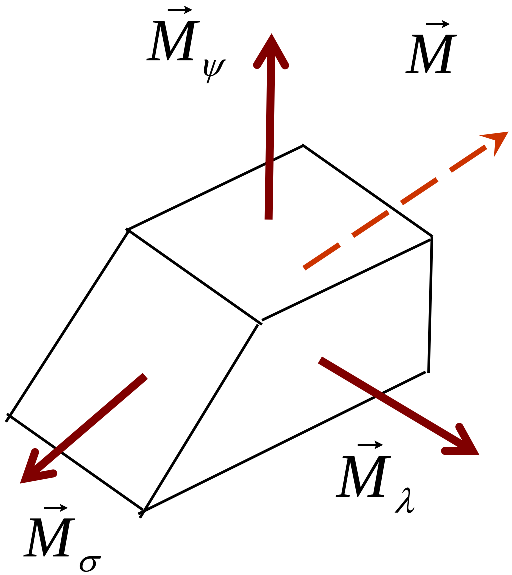
Moment about Point

Moment about line (*projection effect*)

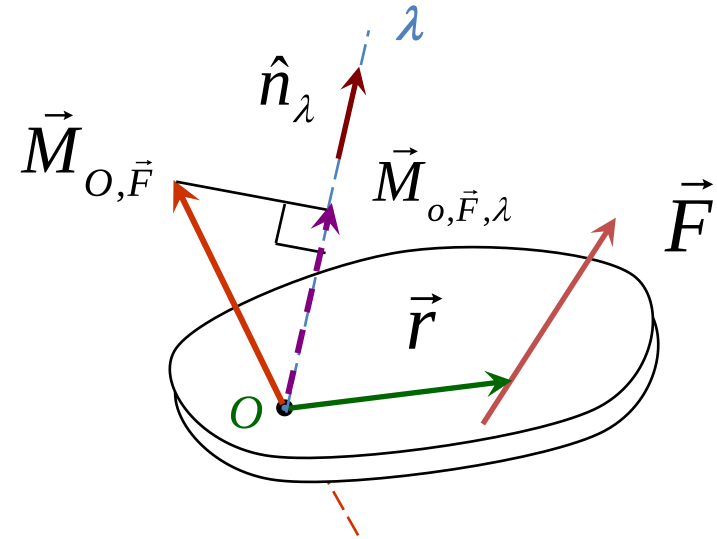
$$\vec{M}_{A,\vec{F}} = \vec{r} \times \vec{F}$$

$$\vec{M}_{A,\vec{F},\lambda_{AB}} = (\vec{M}_{A,\vec{F}} \cdot \hat{e}_{AB})\,\hat{e}_{AB}$$

Finding moment of force about (arbitary) axis λ



$$\vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



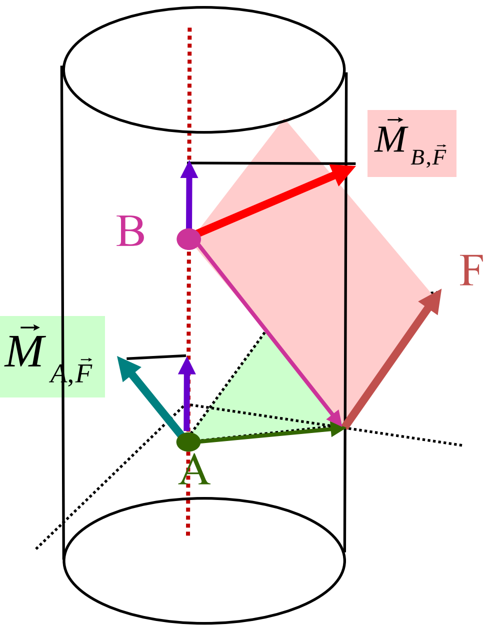
$$\boxed{\vec{M}_{o,\vec{F},\lambda} = ((\vec{r}\times\vec{F})\cdot\hat{n}_{\lambda})\hat{n}_{\lambda}}$$

Depend on line λ only,
Not depend on point O

$$(\vec{r}\times\vec{F}\cdot\hat{n}) = \begin{vmatrix} \alpha & \beta & \gamma \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} r_x & r_y & r_z \\ F_x & F_y & F_z \\ \alpha & \beta & \gamma \end{vmatrix}$$

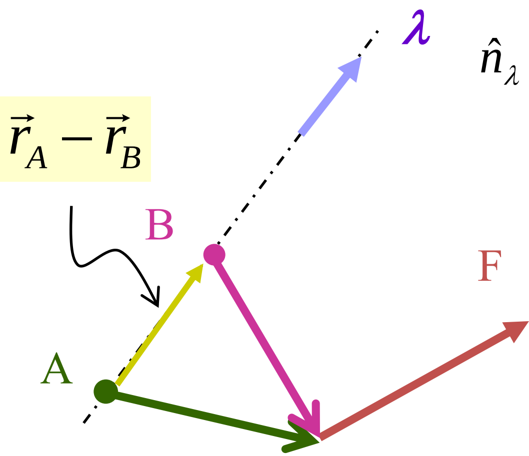
$\alpha,\ \beta,\ \gamma$ are the directional cosines of the unit vector $\hat{n}_{\lambda}$

(i.e. $\hat{n}_{\lambda} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$)



$$\vec{C} = \vec{A} \times \vec{B}$$

$\vec{C} \perp \vec{A}$ and $\vec{C} \perp \vec{B}$



$$(\vec{r}_A - \vec{r}_B) \times \vec{F} \;\perp\; (\vec{r}_A - \vec{r}_B)$$

$$\Rightarrow (\vec{r}_A - \vec{r}_B) \times \vec{F} \;\perp\; \hat{n}_\lambda$$

$\vec{M}_{A,\vec{F}} \neq \vec{M}_{B,\vec{F}}$ (generally)

$\vec{M}_{A,\vec{F},\lambda} = \vec{M}_{B,\vec{F},\lambda}$ where A, B on line λ

Moment of $\vec{F}$ about point {A,B}
in the direction of λ

$$M_{A,\vec{F},\lambda} = (\vec{r}_A \times \vec{F}) \cdot \hat{n}_\lambda$$
$$= (\{(\vec{r}_A - \vec{r}_B) + \vec{r}_B\} \times \vec{F}) \cdot \hat{n}_\lambda$$
$$= ((\vec{r}_A - \vec{r}_B) \times \vec{F} + \vec{r}_B \times \vec{F}) \cdot \hat{n}_\lambda$$
$$= ((\vec{r}_A - \vec{r}_B) \times \vec{F}) \cdot \hat{n}_\lambda + (\vec{r}_B \times \vec{F}) \cdot \hat{n}_\lambda$$

$= \vec{0}$ $\qquad$ $= \vec{M}_{B,\vec{F},\lambda}$

Moment of $\vec{F}$ in the direction of λ
Moment of $\vec{F}$ projected to line λ

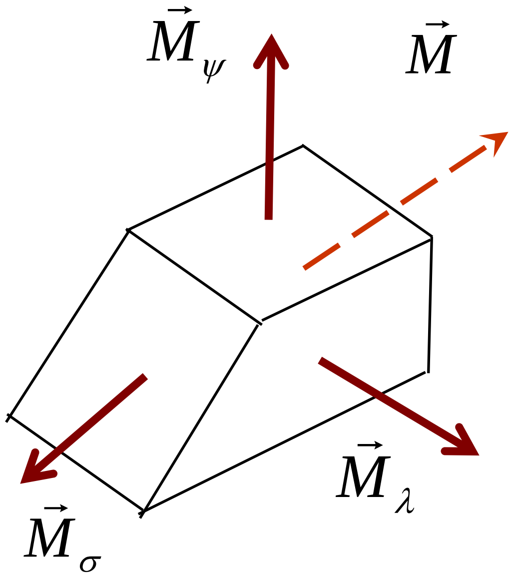
Moment of $\vec{F}$ about line λ

$$\vec{M}_{A,\vec{F},\lambda} = \vec{M}_{B,\vec{F},\lambda} = \ldots \;\square\; \vec{M}_{\vec{F},\lambda}$$

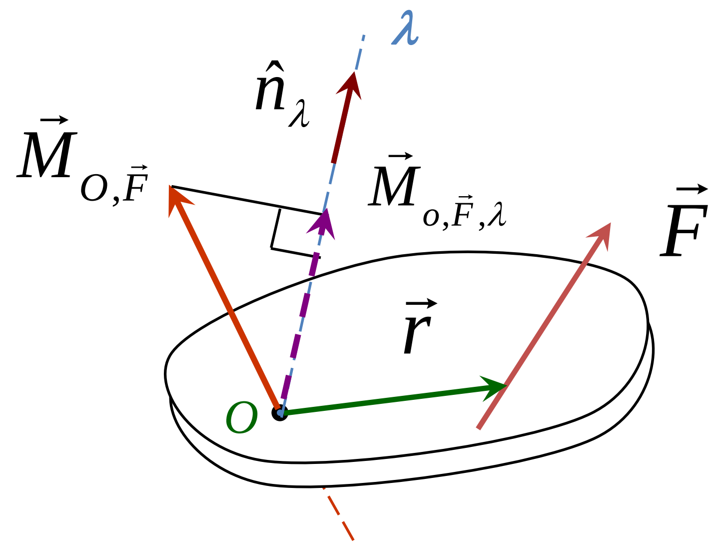
where A, B are any points on the line λ

Moment about axis is sliding vector.

Finding moment of force about (arbitary) axis λ



$$\vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



$$\boxed{\vec{M}_{o,\vec{F},\lambda} = ((\vec{r}\times\vec{F})\cdot\hat{n}_{\lambda})\hat{n}_{\lambda}}$$

Depend on line λ only,
Not depend on point O

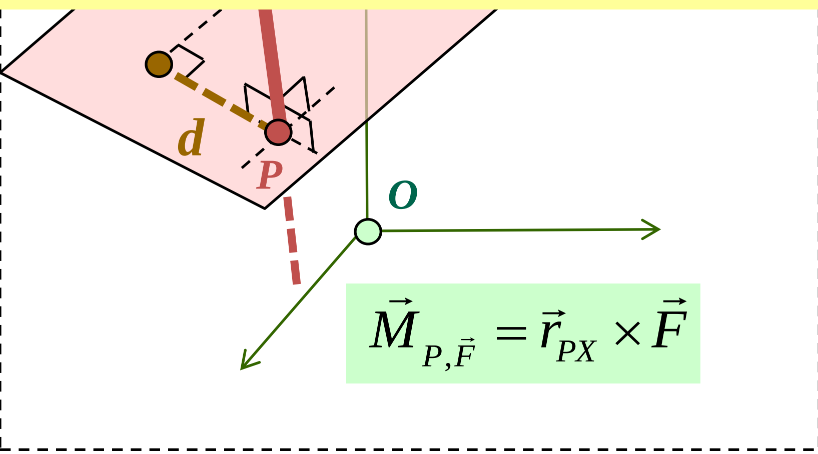
$$(\vec{r}\times\vec{F}\cdot\hat{n}) = \begin{vmatrix} \alpha & \beta & \gamma \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} r_x & r_y & r_z \\ F_x & F_y & F_z \\ \alpha & \beta & \gamma \end{vmatrix}$$

$\alpha,\ \beta,\ \gamma$ are the directional cosines of the unit vector $\hat{n}_{\lambda}$

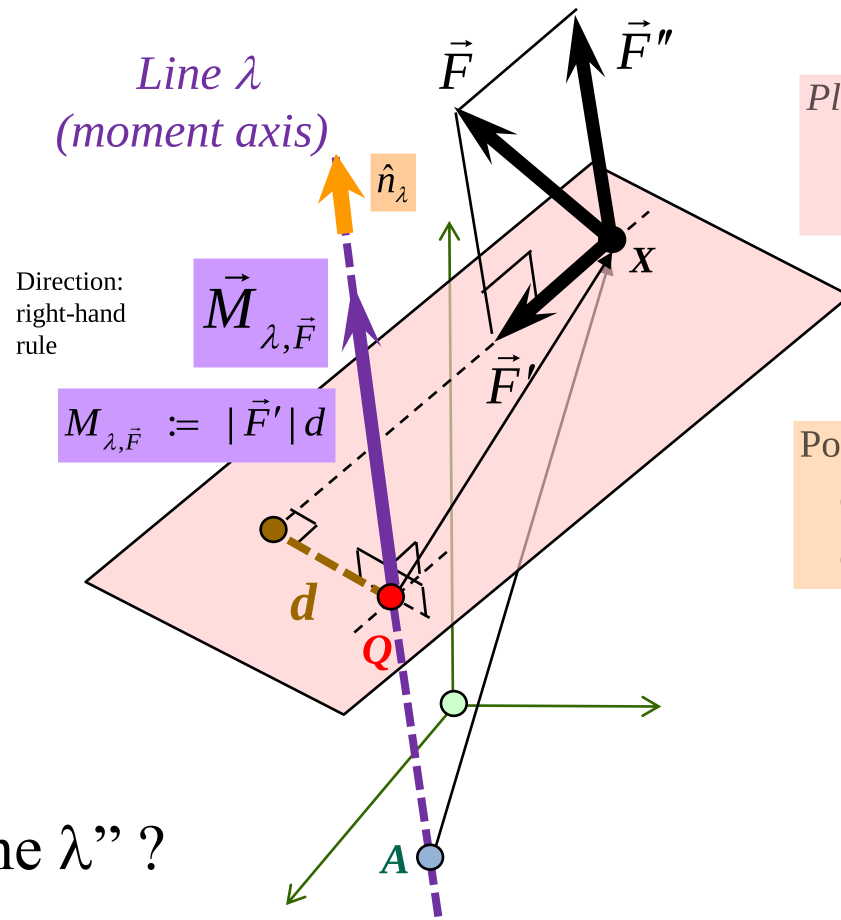
(i.e. $\hat{n}_{\lambda} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$)

Moment about line λ (Definition)

$$\left[\vec{r}_{QX}\times\vec{F}\right]\cdot\hat{n}_\lambda$$
$$=\left[\vec{r}_{QX}\times(\vec{F}'+\vec{F}'')\right]\cdot\hat{n}_\lambda$$
$$=\left[\vec{r}_{QX}\times\vec{F}'+\vec{r}_{QX}\times\vec{F}''\right]\cdot\hat{n}_\lambda$$
$$=\left[\vec{r}_{QX}\times\vec{F}'\right]\cdot\hat{n}_\lambda+\cancel{\left[\vec{r}_{QX}\times\vec{F}''\right]\cdot\hat{n}_\lambda}$$



$$\vec{M}_{P,\vec{F}}=\vec{r}_{PX}\times\vec{F}$$



Plane :
Plane $\perp$ $\hat{n}_\lambda$
X $\subset$ Plane

Point Q :
Q $\subset$ Line λ
Q $\subset$ Plane

A : Any point on line λ

How to find "Moment about line λ" ?

$$M_{\lambda,\vec{F}}=|\vec{F}'|\,d$$

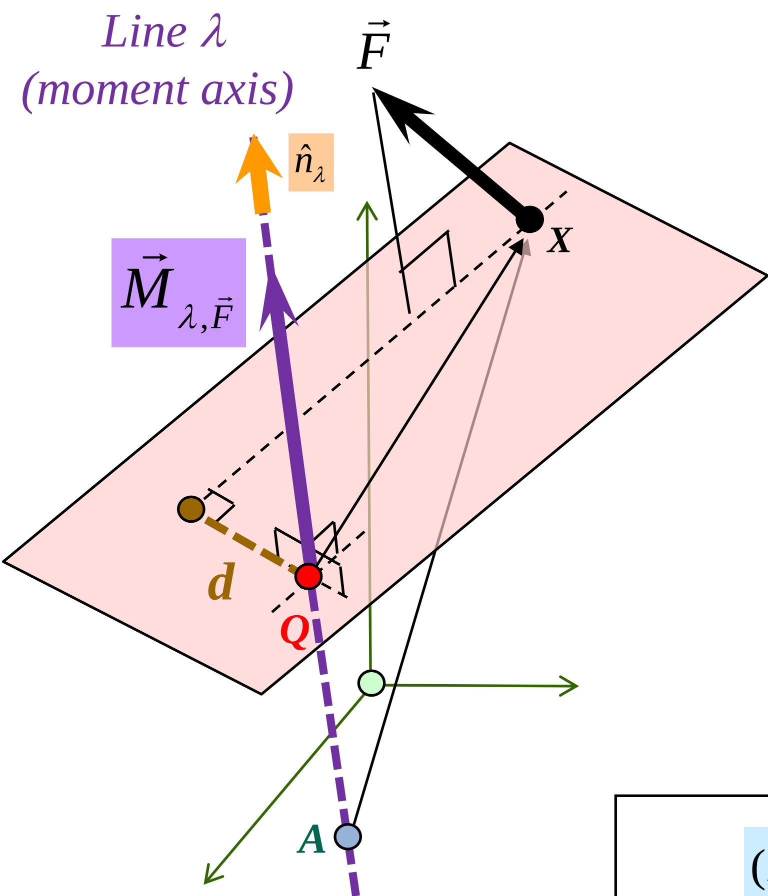
Hard to find

$$=\vec{M}_{A,\vec{F}}\cdot\hat{n}_\lambda$$

$$M_{\lambda,\vec{F}}=|\vec{r}_{QX}\times\vec{F}'|=\left[\vec{r}_{QX}\times\vec{F}'\right]\cdot\hat{n}_\lambda=\left[\vec{r}_{QX}\times\vec{F}\right]\cdot\hat{n}_\lambda=\vec{M}_{Q,\vec{F}}\cdot\hat{n}_\lambda$$

Hard to find

Moment about line λ



We will prove that

$$(\vec{M}_{A,\vec{F}} \cdot \hat{n}_\lambda =)\ M_{A,\vec{F},\lambda}$$

A : Any point on line λ

is equal to

$$M_{Q,\vec{F},\lambda} \left(= \vec{M}_{Q,\vec{F}} \cdot \hat{n}_\lambda \right)$$

Moment of $\vec{F}$ about point {A,Q} projected to line λ

$$M_{A,\vec{F},\lambda} = \left[\vec{r}_{AX} \times \vec{F} \right] \cdot \hat{n}_\lambda$$

$$= \left[(\vec{r}_{AQ} + \vec{r}_{QX}) \times \vec{F} \right] \cdot \hat{n}_\lambda$$

$$= \left[\vec{r}_{AQ} \times \vec{F} + \vec{r}_{QX} \times \vec{F} \right] \cdot \hat{n}_\lambda$$

$$= \underbrace{(\vec{r}_{AQ} \times \vec{F}) \cdot \hat{n}_\lambda}_{\text{must prove to be } 0} + \underbrace{(\vec{r}_{QX} \times \vec{F}) \cdot \hat{n}_\lambda}_{M_{Q,\vec{F},\lambda}}$$

$$(\vec{A} \times \vec{B}) \perp \vec{A}$$

$$(\vec{r}_{AQ} \times \vec{F}) \perp \vec{r}_{AQ}$$

$$\vec{r}_{AQ} \ // \ \hat{n}_\lambda$$

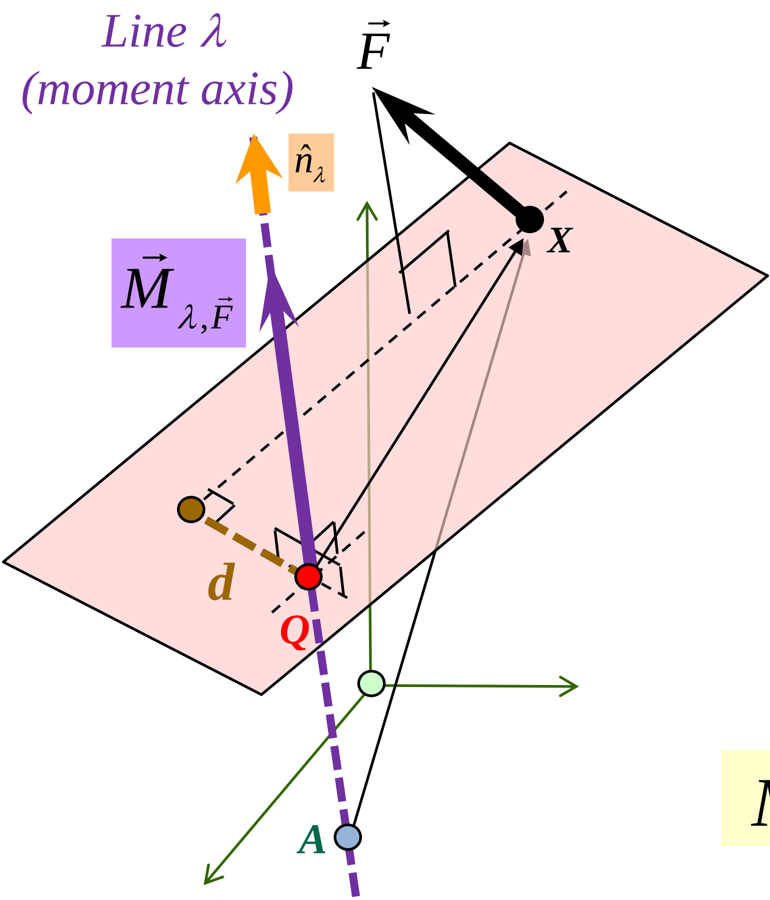
$$\Rightarrow \vec{r}_{AQ} \times \vec{F} \perp \hat{n}_\lambda$$

$$M_{\lambda,\vec{F}} = \vec{M}_{Q,\vec{F}} \cdot \hat{n}_\lambda$$

$$= \vec{M}_{A,\vec{F}} \cdot \hat{n}_\lambda$$

A : Any point on line λ

Moment about line λ



$$(\vec{M}_{A,\vec{F}} \cdot \hat{n}_\lambda =) \quad M_{A,\vec{F},\lambda}$$

is equal to $$M_{Q,\vec{F},\lambda} \quad \left(= \vec{M}_{Q,\vec{F}} \cdot \hat{n}_\lambda\right)$$

Point A is <u>*any*</u> point in the line λ

Moment about axis is sliding vector.

$$\vec{M}_{A,\vec{F},\lambda} = \vec{M}_{B,\vec{F},\lambda} = \ \ldots \quad \ \vec{M}_{\lambda,\vec{F}}$$

where A, B are any points on the line λ

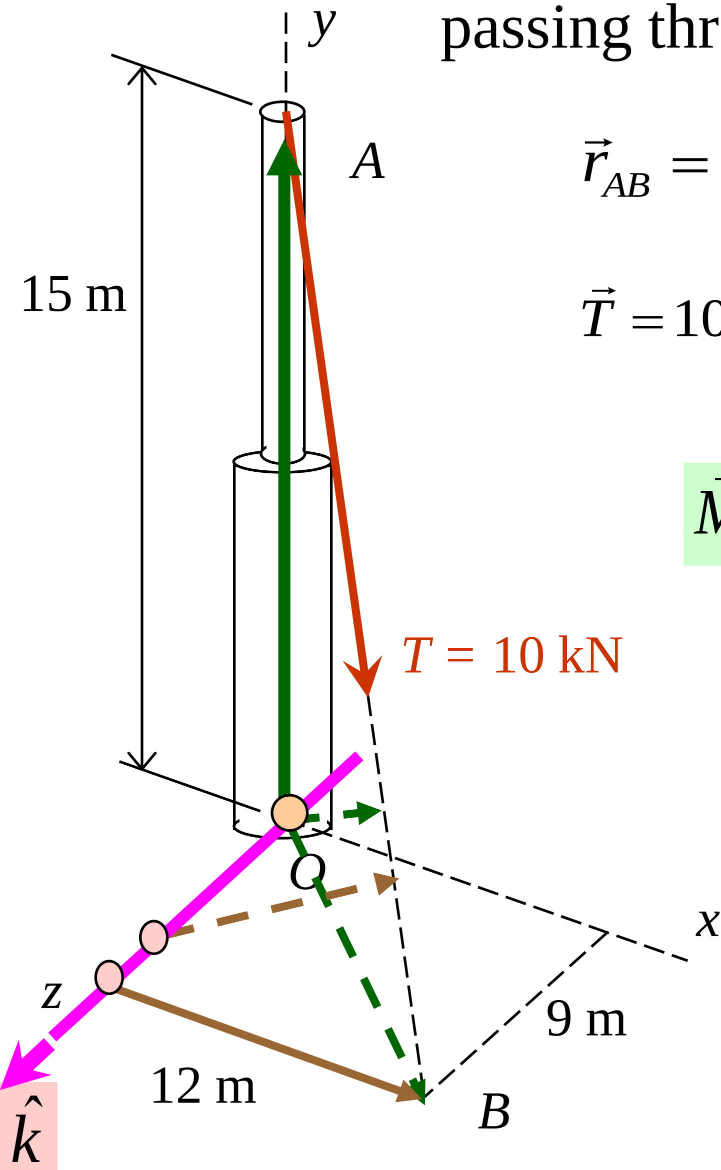
Moment of $\vec{F}$
about line λ

Moment of $\vec{F}$
about point A
in the direction of λ

Moment of $\vec{F}$ in the direction of λ

Moment of $\vec{F}$ projected to line λ

Find $\vec{M}_z$ of $\vec{T}$ (the moment of $\vec{T}$ about z-axis passing through the base O)



$$\vec{r}_{AB} = -15\hat{j} + 12\hat{i} + 9\hat{k} \qquad A(0,15,0) \quad B(12,0,9)$$

$$\vec{T} = 10\frac{12\hat{i} - 15\hat{j} + 9\hat{k}}{\sqrt{12^2 + 15^2 + 9^2}} \text{ kN}$$

$$\vec{M}_o = \vec{r} \times \vec{T} \quad = 15\hat{j} \times 10(0.566\hat{i} - 0.707\hat{j} + 0.424\hat{k})$$

$$= 150(-0.566\hat{k} + 0.424\hat{i}) \text{ kN-m}$$

$$M_z = (\vec{M}_o \cdot \hat{k}) = -84.9 \text{ kN-m}$$

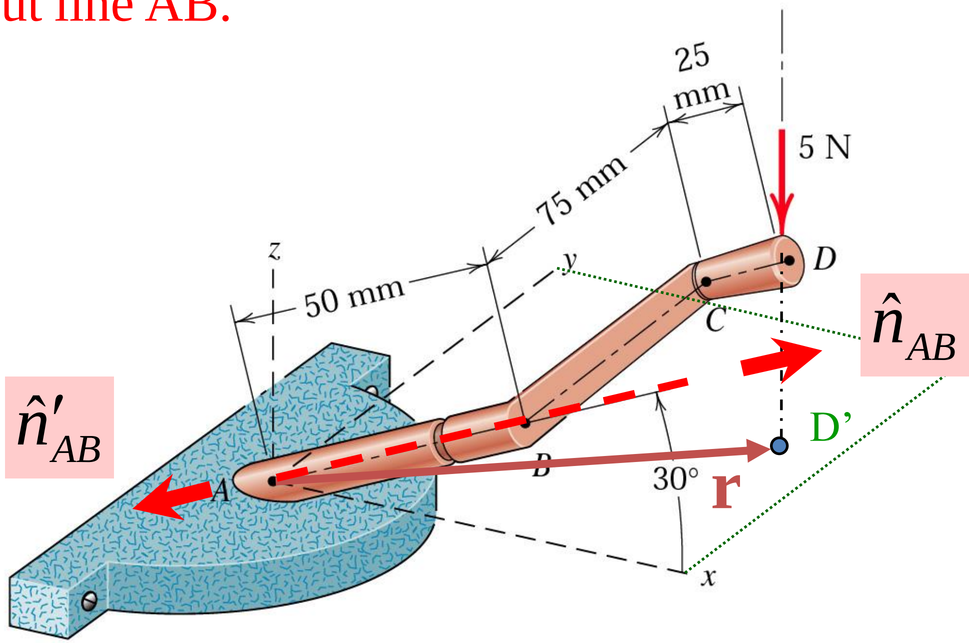
$$\vec{M}_z = (\vec{M}_o \cdot \hat{k})\,\hat{k} = -84.9\hat{k} \text{ kN-m}$$ ***Ans***

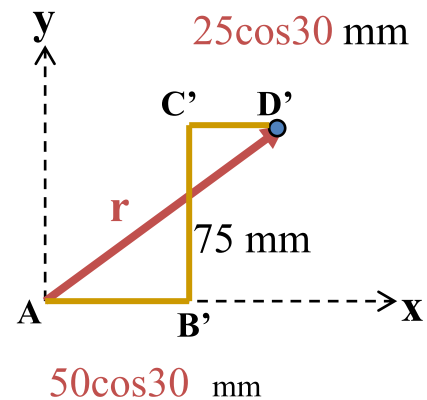
Figure must be shown

$$\vec{M}_{z,\vec{F}} = \{ (\vec{r} \times \vec{T}) \cdot \hat{k} \}\ \hat{k}$$ OK OK

$$\vec{M}_{O,\vec{F}} = (\vec{r} \times \vec{T})$$ OK not OK

2/133 A 5N vertical force is applied to the knob of the window-opener mechanism when the crank BC is horizontal. Determine the moment of force about point A and about line AB.





$$\vec{r}_{AD} = (75\cos 30^o)\hat{i} + 75\,\hat{j}$$

$$\vec{M}_A = \vec{r} \times \vec{F}$$

$$= (75\cos 30\hat{i} + 75\,\hat{j}) \times (-5\hat{k})$$

$$= -375\hat{i} + 325\,\hat{j} \quad \text{N-mm} \quad \textbf{\underline{Ans}}$$

$$\hat{n}_{AB,1} = \cos 30\hat{i} + \sin 30\hat{k}$$

$$M_{AB} = \vec{M}_A \cdot \hat{n}_{AB} = -162.26 \quad \text{N-mm}$$

$$\vec{M}_{AB} = (\vec{M}_A \cdot \hat{n}_{AB})\hat{n}_{AB}$$

$$= -126.26(\cos 30^o\,\hat{i} + \sin 30^o\,\hat{k}) \quad \text{N-mm}$$

Ans

$$\hat{n}'_{AB} = \; -(\cos 30\hat{i} + \sin 30\hat{k})$$

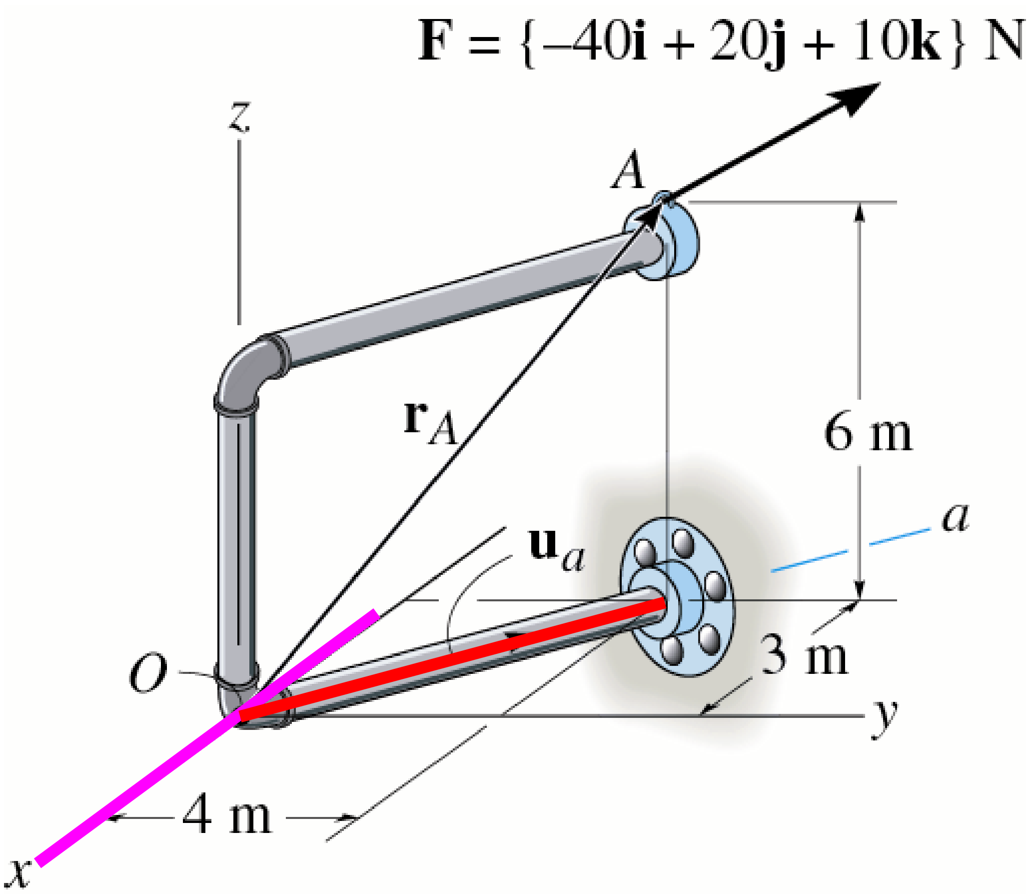
$$M'_{AB} = \vec{M}_A \cdot \hat{n}'_{AB} = +162.26$$

$$\vec{M}_{AB} = (\vec{M}_A \cdot \hat{n}_{AB})\hat{n}_{AB}$$

$$= +126.26\left[-(\cos 30^o\,\hat{i} + \sin 30^o\,\hat{k})\right] \quad \text{N-mm}$$

Example Hibbeler Ex 4-8 #1

Determine the moments of this force about the *x* and *a* axes.



$$\vec{r}_A = (-3\hat{i} + 4\hat{j} + 6\hat{k})\ \text{m}$$

$$\hat{u}_x = \hat{i}$$

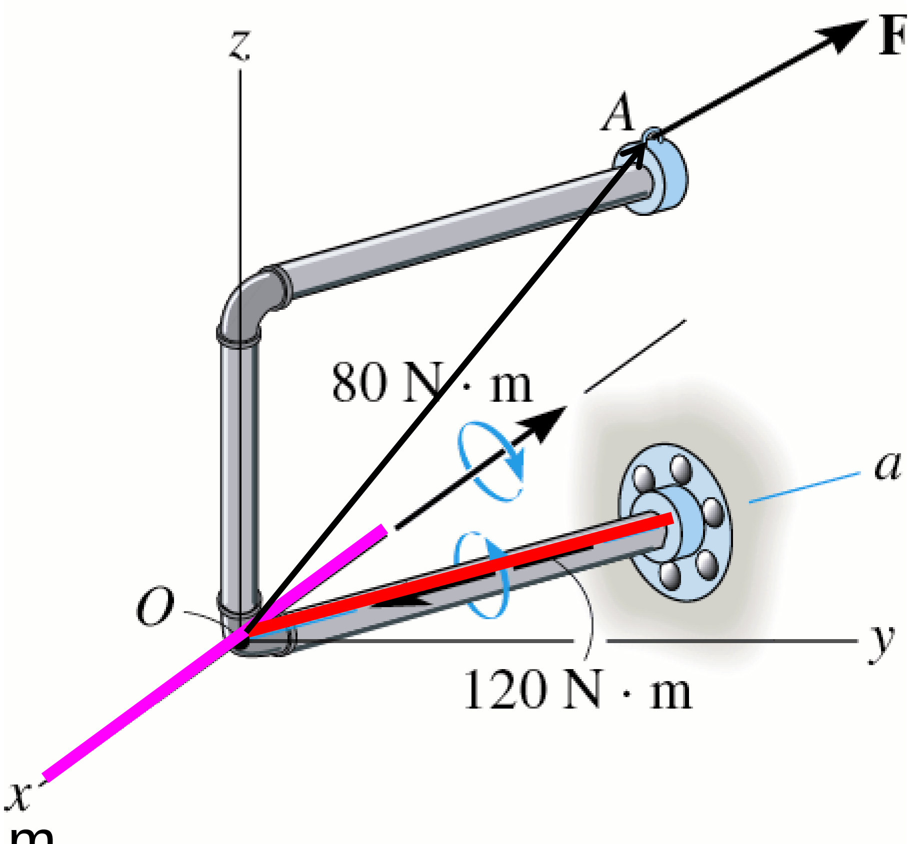
$$M_x = \hat{u}_x \cdot (\vec{r}_A \times \vec{F})$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 6 \\ -40 & 20 & 10 \end{vmatrix}\ \text{N}\cdot\text{m}$$

$$= -80\ \text{N}\cdot\text{m}$$

$$\vec{M}_x = -80\hat{i}\ \text{N}\cdot\text{m} \qquad \#$$

Example Hibbeler Ex 4-8 #2



$$\hat{u}_a = (-\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j})\ \text{m}$$

$$M_a = \hat{u}_a \cdot (\vec{r}_A \times \vec{F})$$

$$= \begin{vmatrix} -3/5 & 4/5 & 0 \\ -3 & 4 & 6 \\ -40 & 20 & 10 \end{vmatrix}\ \text{N}\cdot\text{m}$$

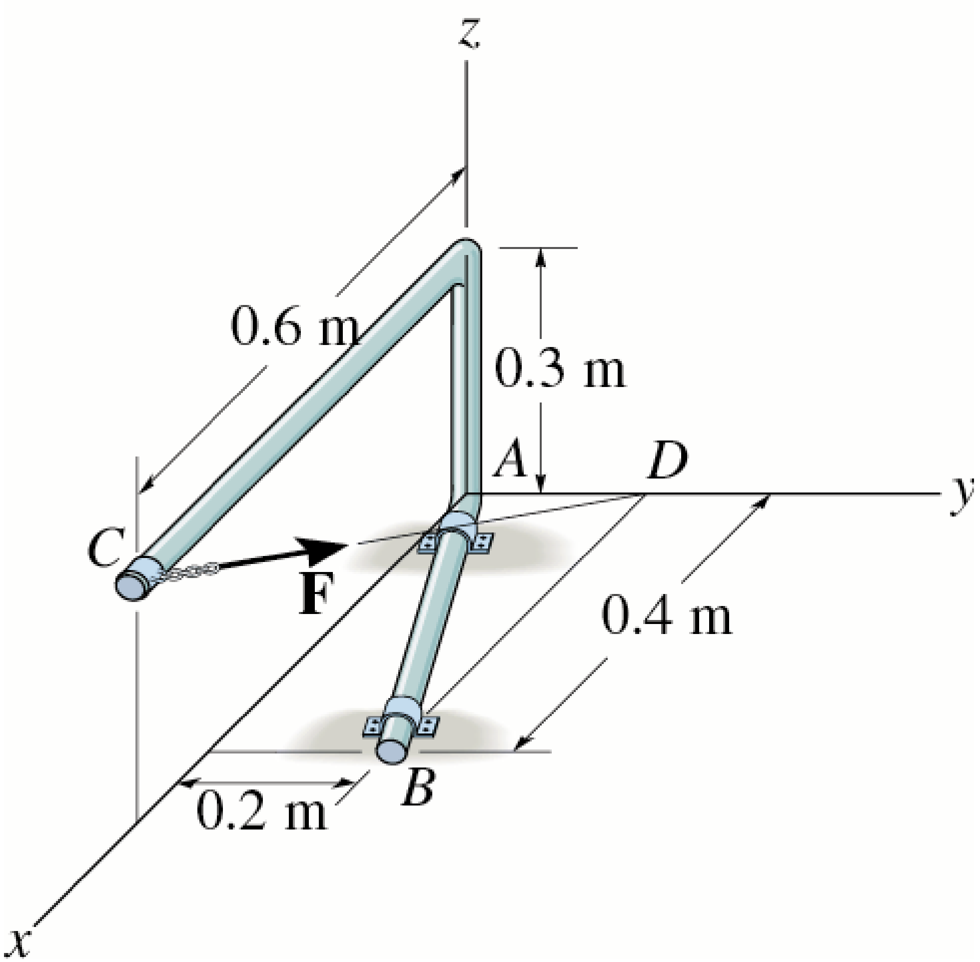
$$= (-\frac{3}{5})\begin{vmatrix} 4 & 6 \\ 20 & 10 \end{vmatrix} - \frac{4}{5}\begin{vmatrix} -3 & 6 \\ -40 & 10 \end{vmatrix}\ \text{N}\cdot\text{m}$$

$$= -120\ \text{N}\cdot\text{m}$$

$$\vec{M}_a = M_a\hat{u}_a = (72\hat{i} - 96\hat{j})\ \text{N}\cdot\text{m}$$ #

Example Hibbeler Ex 4-9 #1

Determine the moment $\mathbf{M}_{AB}$ produced by **F** = (–600**i** + 200**j** – 300**k**) N, which tends to rotate the rod about the *AB* axis.



$$M_{AB} = \hat{u}_B \cdot (\vec{r} \times \vec{F})$$

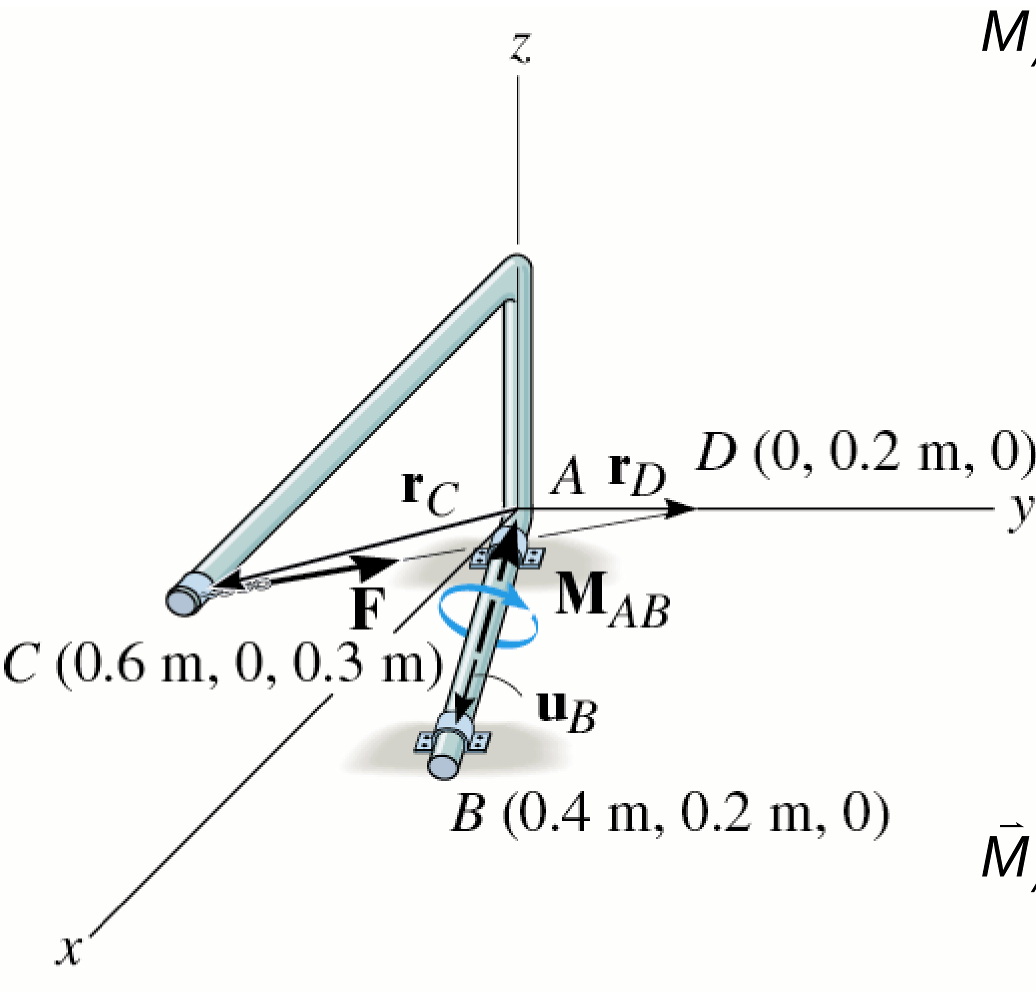
$$\hat{u}_B = \frac{\vec{r}_B}{r_B} = \frac{0.4\hat{i} + 0.2\hat{j}}{\sqrt{0.4^2 + 0.2^2}}$$

$$\hat{u}_B = 0.89443\hat{i} + 0.44721\hat{j}$$

Vector **r** is directed from any point on the *AB* axis to any point on the line of action of the force.

$$\vec{r} = \vec{r}_D = 0.2\hat{j} \text{ m}$$

Example Hibbeler Ex 4-9 #2



$$M_{AB} = \hat{u}_B \cdot (\vec{r}_D \times \vec{F})$$

$$= \begin{vmatrix} 0.89443 & 0.44721 & 0 \\ 0 & 0.2 & 0 \\ -600 & 200 & -300 \end{vmatrix}$$

$$= 0.2 \begin{vmatrix} 0.89443 & 0 \\ -600 & -300 \end{vmatrix}$$

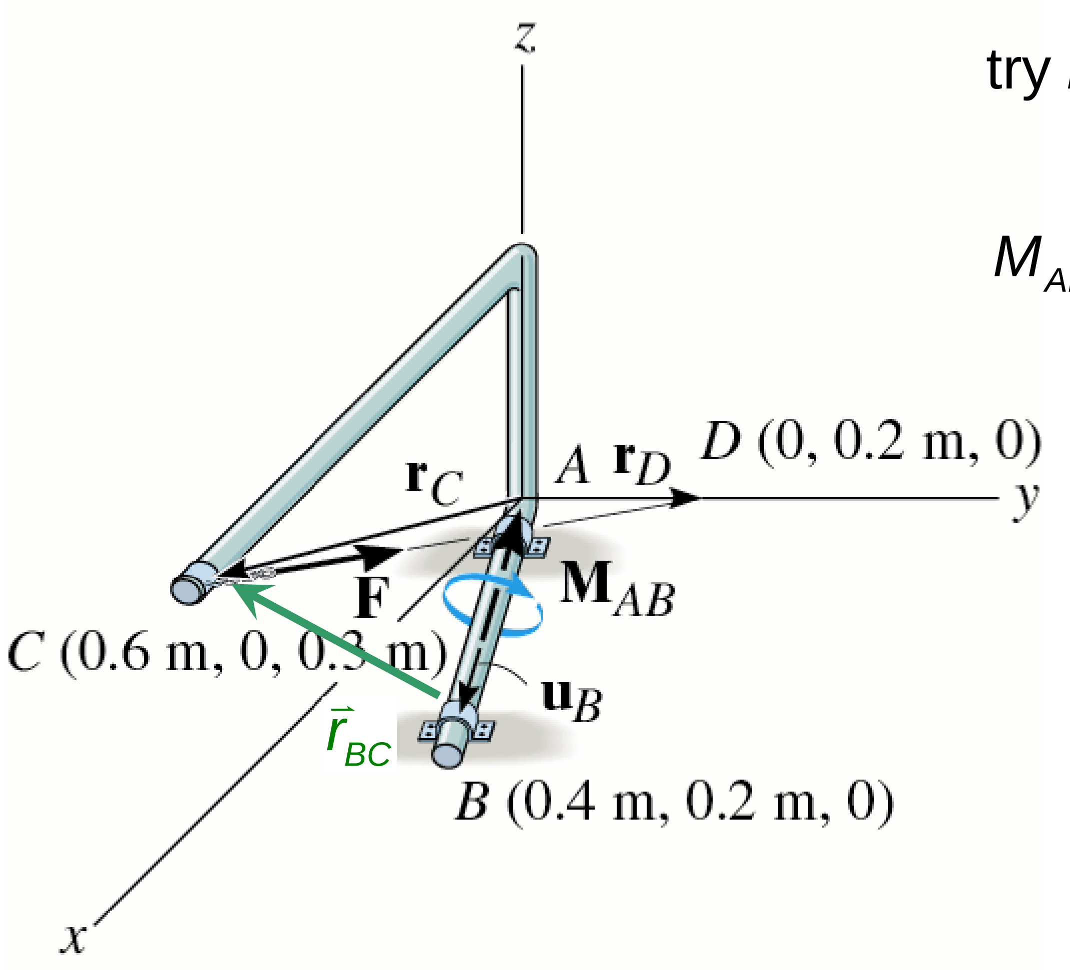
$$= -53.666 \text{ N} \cdot \text{m}$$

$$\vec{M}_{AB} = M_{AB} \hat{u}_B$$

$$= (-48.0\hat{i} - 24.0\hat{j}) \text{ N} \cdot \text{m} \quad \#$$

Example Hibbeler Ex 4-9 #3

Vector **r** is directed from any point on the *AB* axis to any point on the line of action of the force.

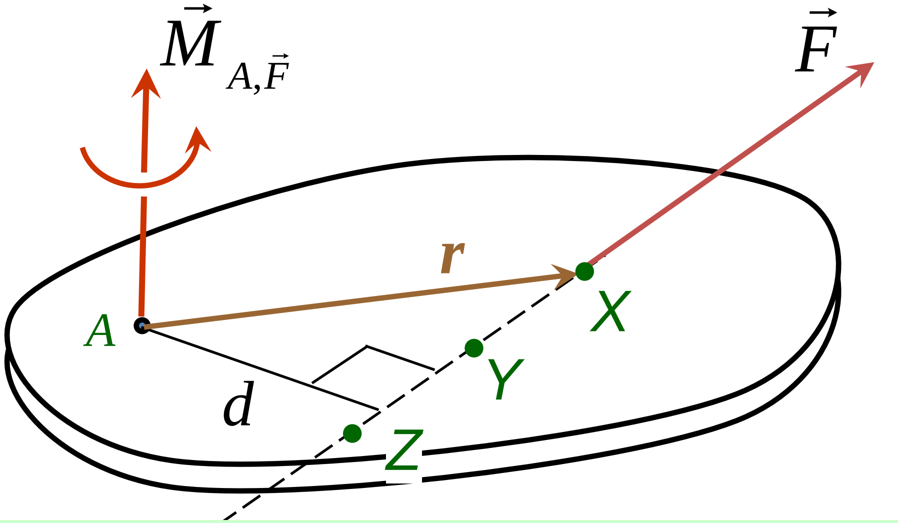


$$\text{try } \vec{r} = \vec{r}_{BC} = 0.2\hat{i} - 0.2\hat{j} + 0.3\hat{k} \text{ m}$$

$$M_{AB} = \hat{u}_B \cdot (\vec{r}_{BC} \times \vec{F})$$

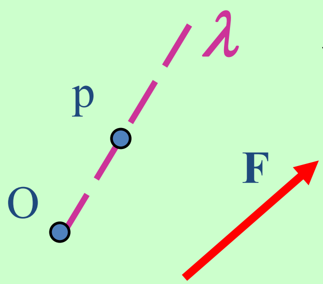
$$= \begin{vmatrix} 0.89443 & 0.44721 & 0 \\ 0.2 & -0.2 & 0.3 \\ -600 & 200 & -300 \end{vmatrix}$$

$$= -53.665 \text{ N} \cdot \text{m}$$



$$\vec{M}_{\vec{F},A} = \vec{r} \times \vec{F}$$

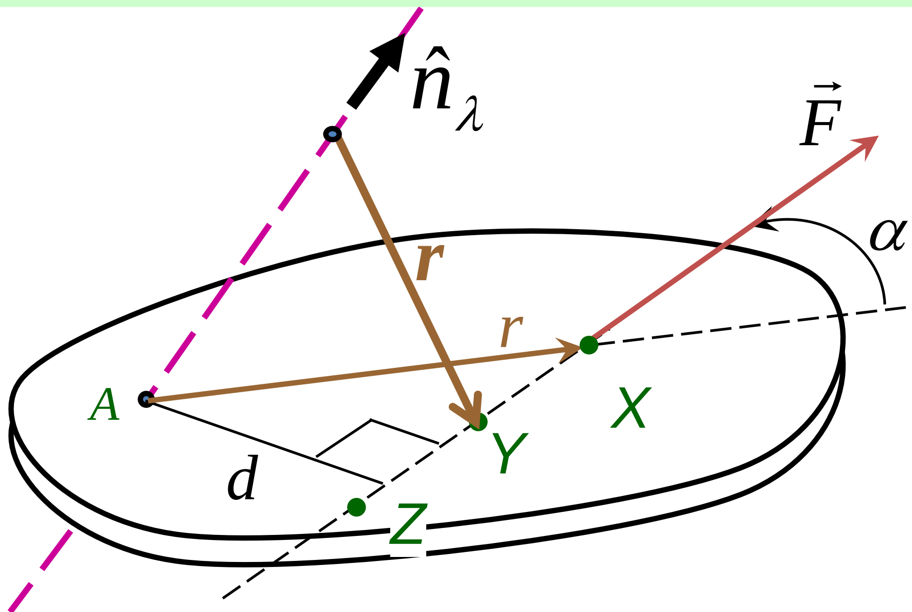
position vector:
from A to point of application of the force



$$\forall p \in line_{\lambda} :$$

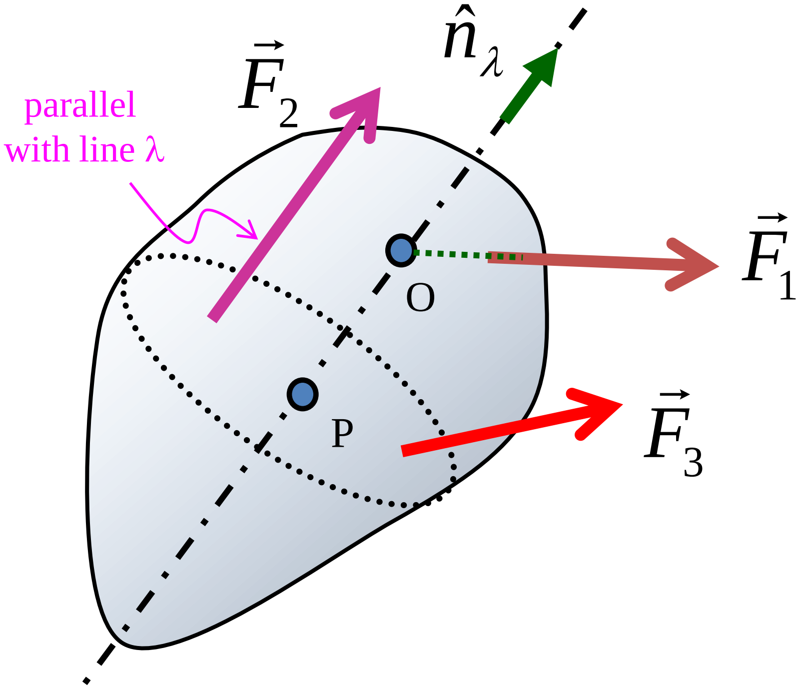
$$\vec{M}_{p,\vec{F},\lambda} = \vec{M}_{o,\vec{F},\lambda}$$

position vector:
from A to any point on line of action of the force.



$$\vec{M}_{\vec{F},\lambda} = \{ (\vec{r} \times \vec{F}) \cdot \hat{n}_{\lambda} \} \hat{n}_{\lambda}$$

position vector:
from any point on line λ to any point on tline of action of the force.



$$\boxed{\sum \vec{M}_\lambda = ?}$$

$$= \left\{ \sum_{i=1}^{3} \{(\vec{r} \times \vec{F}_i) \cdot \hat{n}_\lambda\} \right\} \hat{n}_\lambda$$

$$\boxed{= \{ (\vec{r} \times \vec{F}_3) \cdot \hat{n}_\lambda \} \hat{n}_\lambda}$$

Why?

$\vec{F}_1$ intersects with that axis.

$$M_{\lambda,\, \vec{F}_1} = \left(\vec{r}_{oo} \times \vec{F}_1 \right) \cdot \hat{n}_\lambda = 0$$

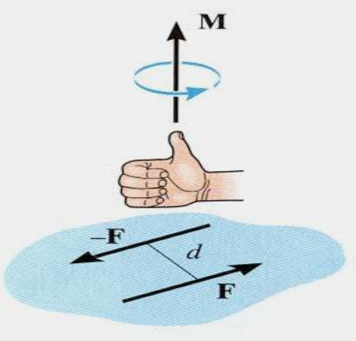
$\vec{F}_2$ is parallel with that axis.

$$\vec{M}_{\mathrm{P},\, \vec{F}_2} \perp \vec{F}_2 \quad \text{and} \quad \vec{F}_2 \, // \, \hat{n}_\lambda$$

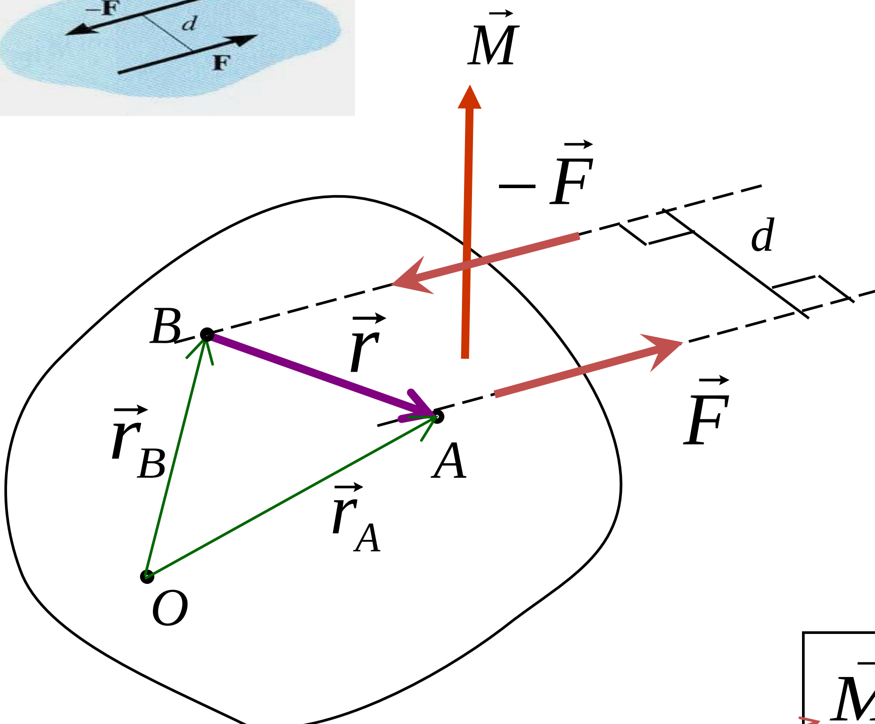
$$\Rightarrow \quad \vec{M}_{P,\, \vec{F}_2} \perp \hat{n}_\lambda$$

Forces which interest or parallel with axis, do not cause the moment about that axis

Couple



Couple is a summed moment produced by two force of equal magnitude but opposite in direction.



$$\vec{M}_O = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F}) = (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

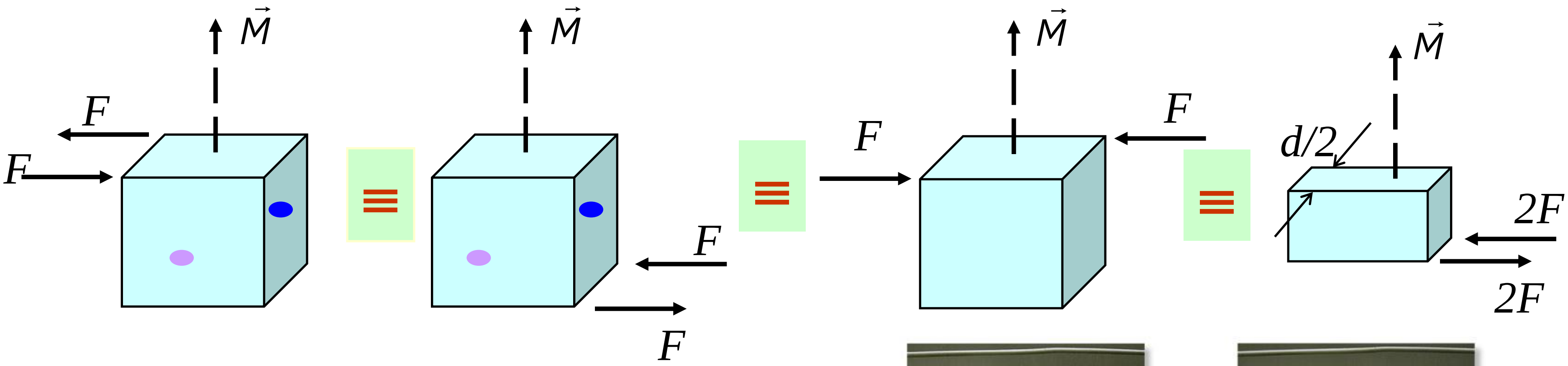
$$\boxed{\vec{M} = \vec{r} \times \vec{F}}$$

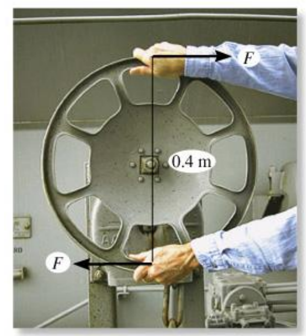
magnitude and direction
Do not depend on O

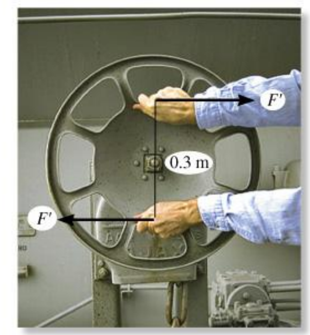
from any point on line of the action
to any point on the other line of action

Moment of a couple is the same about all point
→ Couple may be represented as a free vector.

The followings are equivalent couples





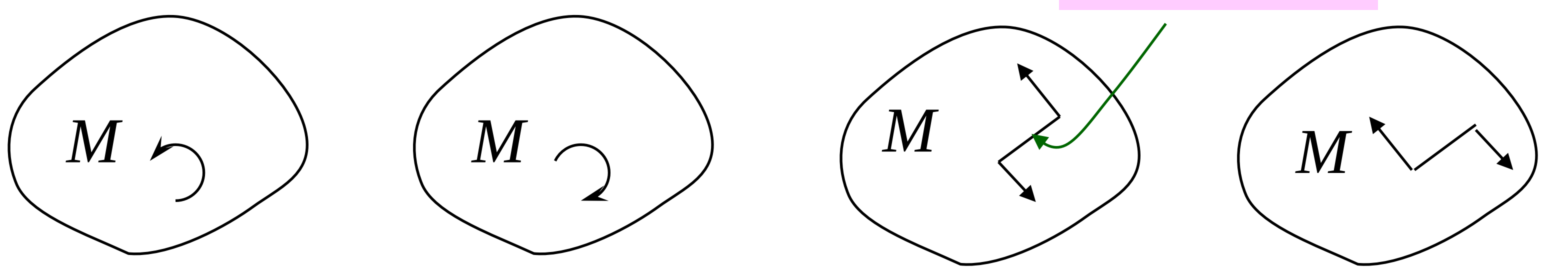


Every point has the equivalent moment.

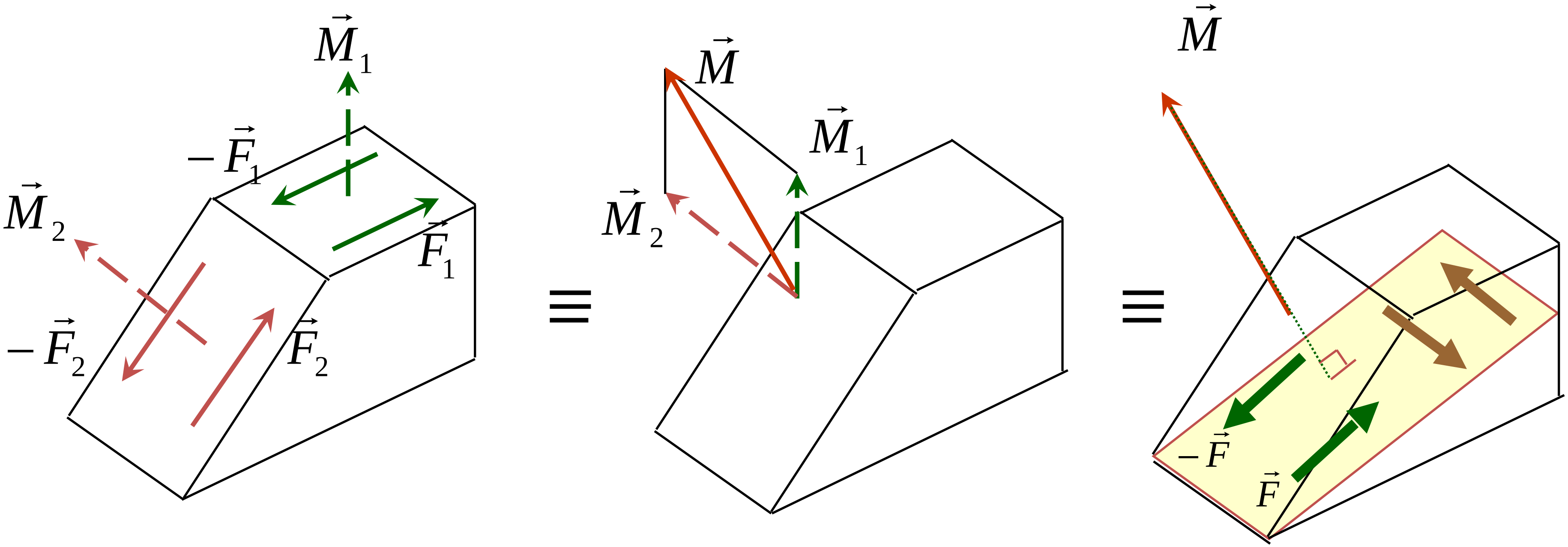
2D representations: (Couples)

couple is

a free vector



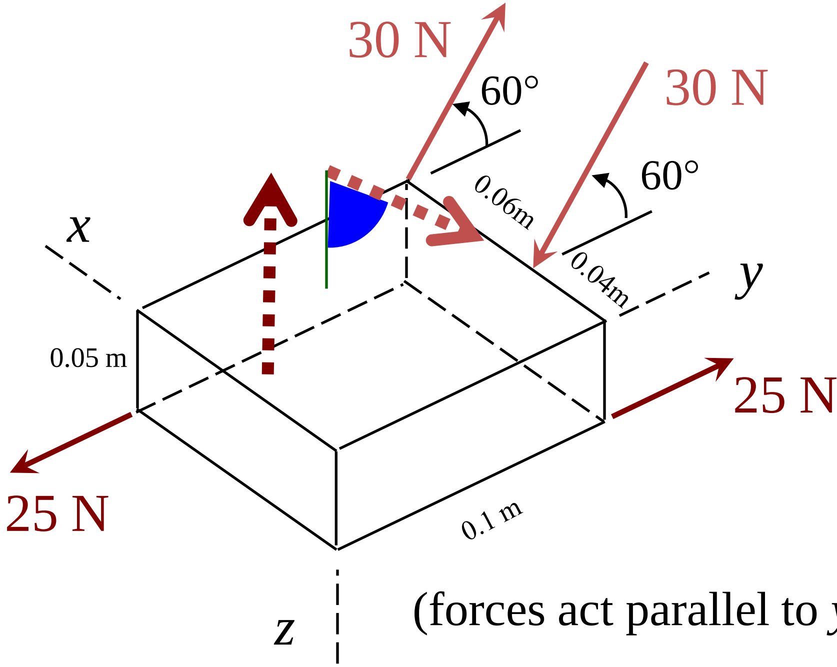
- Couple tends to produce a "pure" rotation of the body about an axis normal to the plane of the forces (which constitute the couple); i.e. the axis of the couple.



- Couples obey all the usual rules that govern vector quantities.

- Again, couples are free vector. After you add them (vectorially), the point of application are not needed!!!

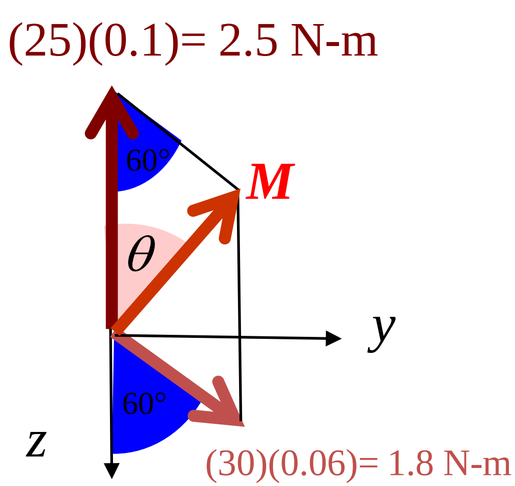
- Compare to adding forces (i.e. finding resultant), after you add the forces vectorially (i.e. obtaining the magnitude and direction of the resultant), you still need to find the line of action of the resultant (2D - 2/6, 3D - 2/9).



(forces act parallel to *y*-*z* plane)

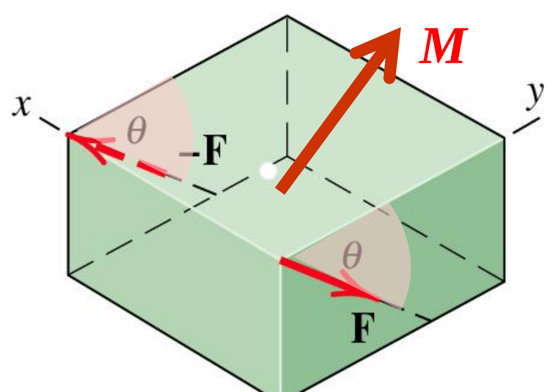
1) Replace the two couples with a single couple $\vec{M}$ that still produces the same external effect on the block.

2) Find two forces $\vec{F}$ and $-\vec{F}$ on two faces of the block that parallel to the *y*-*z* plane that will replace these four forces.



$$M = \sqrt{1.8^2 + 2.5^2 - 2(1.8)(2.5)\cos 60^o} = 2.23 \text{ N-m}$$

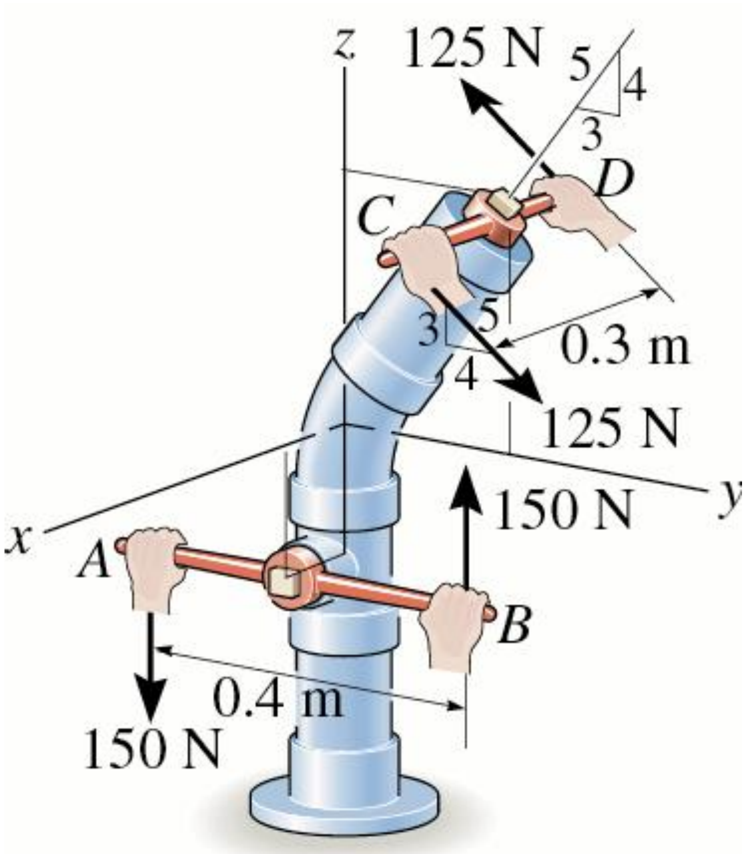
$$\frac{1.8}{\sin\theta} = \frac{2.23}{\sin 60^o} \Rightarrow \theta = 44.3^o$$



$$F = \frac{M}{d} = \frac{2.23}{0.10} = 22.3 \quad \text{N}$$

Example Hibbeler Ex 4-13 #1

Replace the two couples acting on the pipe column by a resultant couple moment.



$$M_1 = d \cdot F = (150\ \text{N})(0.4\ \text{m}) = 60\ \text{N} \cdot \text{m}$$

$$\vec{M}_1 = (60\hat{i})\ \text{N} \cdot \text{m}$$

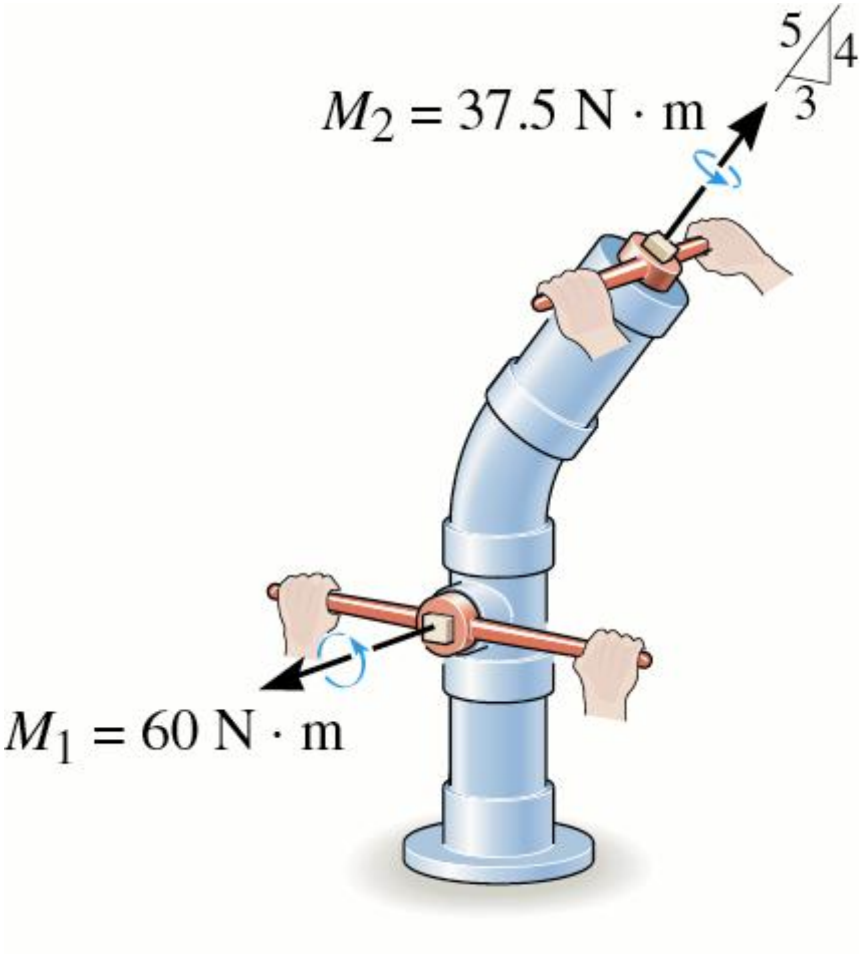
$$\vec{M}_2 = \vec{r}_{DC} \times \vec{F}_C$$

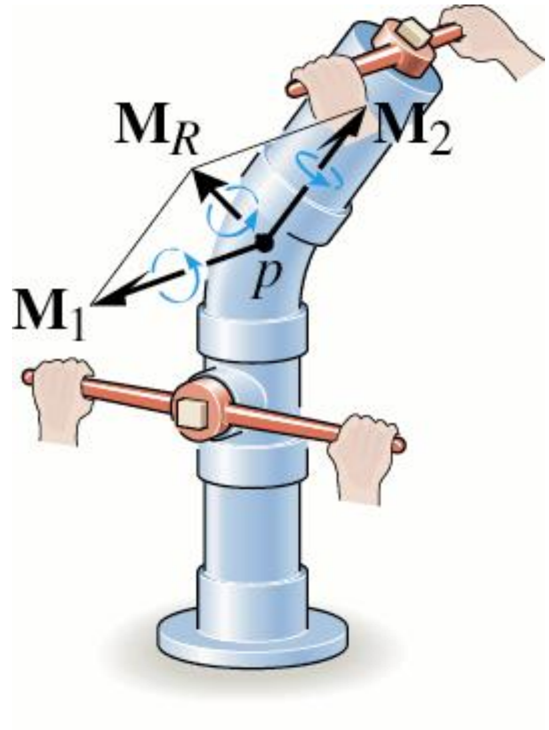
$$= (0.3\hat{i}) \times 125 \left[\frac{4}{5}\hat{j} - \frac{3}{5}\hat{k} \right] \text{N} \cdot \text{m}$$

$$= \left[30(\hat{i} \times \hat{j}) - 22.5(\hat{i} \times \hat{k}) \right] \text{N} \cdot \text{m}$$

$$= (22.5\hat{j} + 30\hat{k})\ \text{N} \cdot \text{m}$$

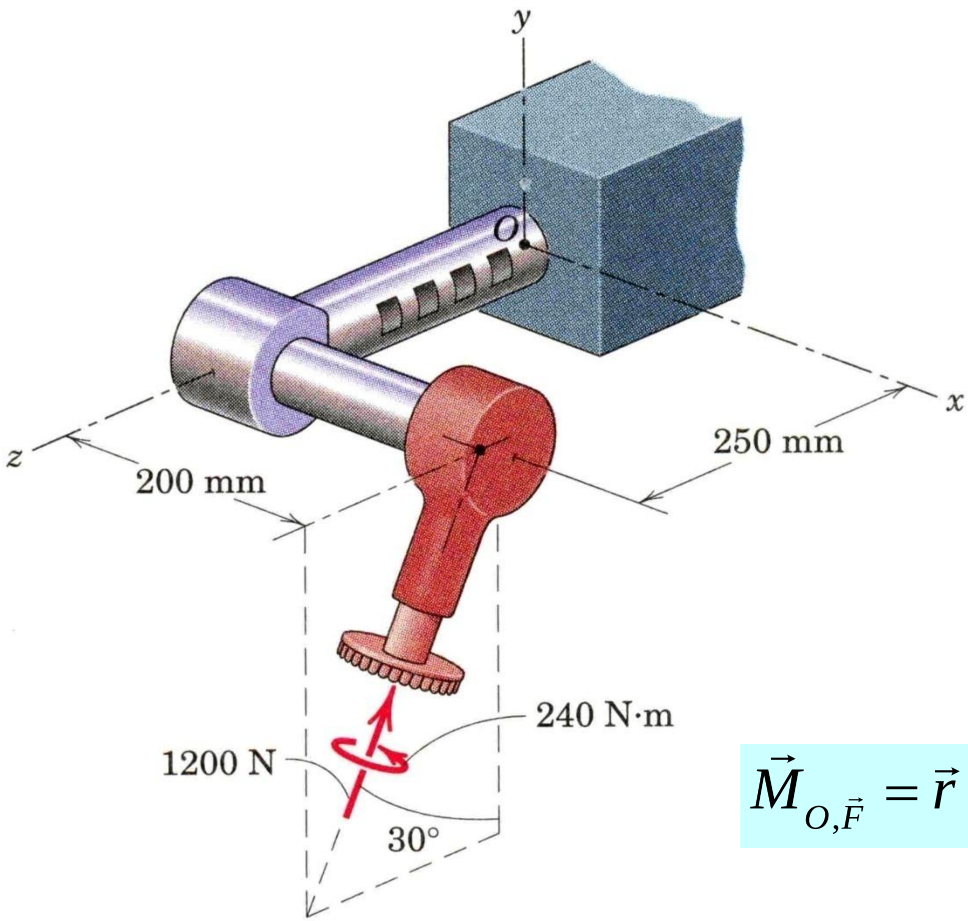
Example Hibbeler Ex 4-13 #2



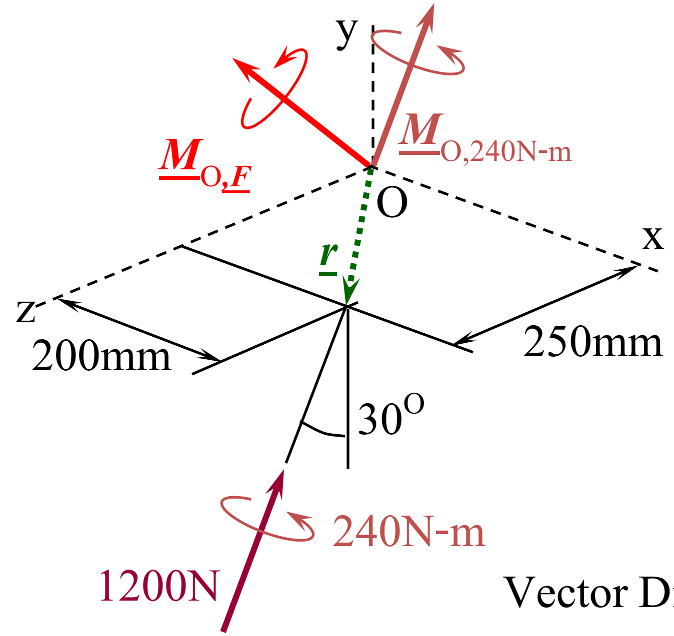


$$\vec{M}_R = \vec{M}_1 + \vec{M}_2 = (60\hat{i} + 22.5\hat{j} + 30\hat{k})\ \mathrm{N \cdot m} \quad \#$$

2/130 The special-purpose milling cutter is subjected to the force of 1200 N and a couple of 240 N·m as shown. Determine the moment of this system about point O.



Problem 2/130



Vector Diagram

$$\vec{F} = 1200(\cos 30^o\,\hat{j} - \sin 30^o\,\hat{k})$$

$$\vec{r} = 0.2\hat{i} + 0.25\hat{k}$$

$$\boxed{\vec{M}_{O,\vec{F}} = \vec{r}\times\vec{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.2 & 0 & 0.25 \\ 0 & 1039 & -600 \end{vmatrix} = -260\hat{i} + 120\hat{j} + 208\hat{k}$$

$$\vec{M}_{O,\ 240\text{N-m}} = 240\left(\cos 30^o\,\hat{j} - \sin 30^o\,\hat{k}\right)$$

$$\boxed{\vec{M}_O = \vec{M}_{O,\vec{F}} + \vec{M}_{O,240N-m} = -260\hat{i} + 328\hat{j} + 88\hat{k}}$$ N-m ***Ans***

Concepts #1

- Vectors can be manipulated by ***scalar multiplication***, ***addition***, subtraction, ***dot product***, ***cross product*** and mixed triple product. Vectors representing can be classified into free, sliding and fixed vectors.
- Position vectors describe the position of a point ***relative to a reference point*** or the origin.
- Statically, force is the ***action of one body on another***. In dynamics, force is an action that tends to cause acceleration of an object. To define a force on ***rigid bodies***, the ***magnitude***, ***direction*** and ***line of action*** are required. Thus, the ***principle of transmissibility*** is applicable to forces on rigid bodies.

Concepts #2

- To define a ***moment about a point***, the magnitude, direction and the point are required. To define a ***moment about an axis***, the magnitude, direction and the axes are required. To define a ***couple***, the magnitude and direction are required.

Chapter Objectives Descriptions #1

- Use mathematical formulae to manipulate physical quantities
  - Specify idealized vector quantities in real worlds and vice versa
  - Obtain magnitude, direction and position of a vector
  - Manipulate vectors by scalar multiplication, addition, subtraction, dot product, cross product and mixed triple product
  - Describe the physical meanings of vector manipulations
- Obtain position vectors with appropriate representation.

Chapter Objectives Descriptions #2

- Use and manipulate force vectors
  - Identify and categorize force vectors
  - Describe the differences between force representation in rigid and deformable bodies
  - Identify and represent forces in real worlds with sufficient data and vice versa
  - Manipulate force vectors

Chapter Objectives Descriptions #3

- Use and manipulate moment vectors
  - Identify and categorize moment vectors
  - Describe the differences between moments about points, moments about axes and couple
  - Identify and represent moments in real worlds with sufficient data and vice versa
  - Manipulate moment vectors

Review Quiz #1

- Use mathematical formulae to manipulate physical quantities
  - Give 4 examples of vector quantities in real world.
  - In how many ways can we specify a 2D/3D vector? Describe each of them.
  - How can we prove that two vectors are parallel?
  - What are the differences between the vector additions by the parallelogram and triangular constructions?
  - Even though we can manipulate vectors analytically, why do we still learn the graphical methods?

Review Quiz #2

- Use mathematical formulae to manipulate physical quantities
  - What are the mathematical definitions of dot, cross and mixed triple products?
  - What are the physical meanings of addition, subtraction, dot product, cross product and mixed triple product?
  - What are the meanings of associative, distributive and commutative properties of products?
  - What are the differences between 2D and 3D vector manipulation?

Review Quiz #3

- Obtain position vectors with appropriate representation.
  - Given points *A* and *B*, what information do you need to obtain the position vector and what name will you give to the position vectors and distance vector between the two points?

Review Quiz #4

- Use and manipulate force vectors
  - For the following forces – tension in cables, forces in springs, weight, magnetic force, thrust of rocket engine, what are their classification in the following force types – external/internal, body/surface and concentrated/distributed forces?
  - If a surface is said to be smooth, what does that mean?
  - What are the differences between force representation in rigid and deformable bodies?
  - What are the additional cautions in force vector manipulation that are not required in general vector manipulation?

Review Quiz #5

- Use and manipulate moment vectors
  - Give 5 examples of moments in real world and approximate them into mathematical models.
  - What information do you need to specify a moment?
  - What is the meaning of moment direction?
  - If a force passes through a point *P*, what do you know about the moment of the force about *P*?
  - What are the differences between physical meanings of moments about points, moments about axis and couples?

Review Quiz #6

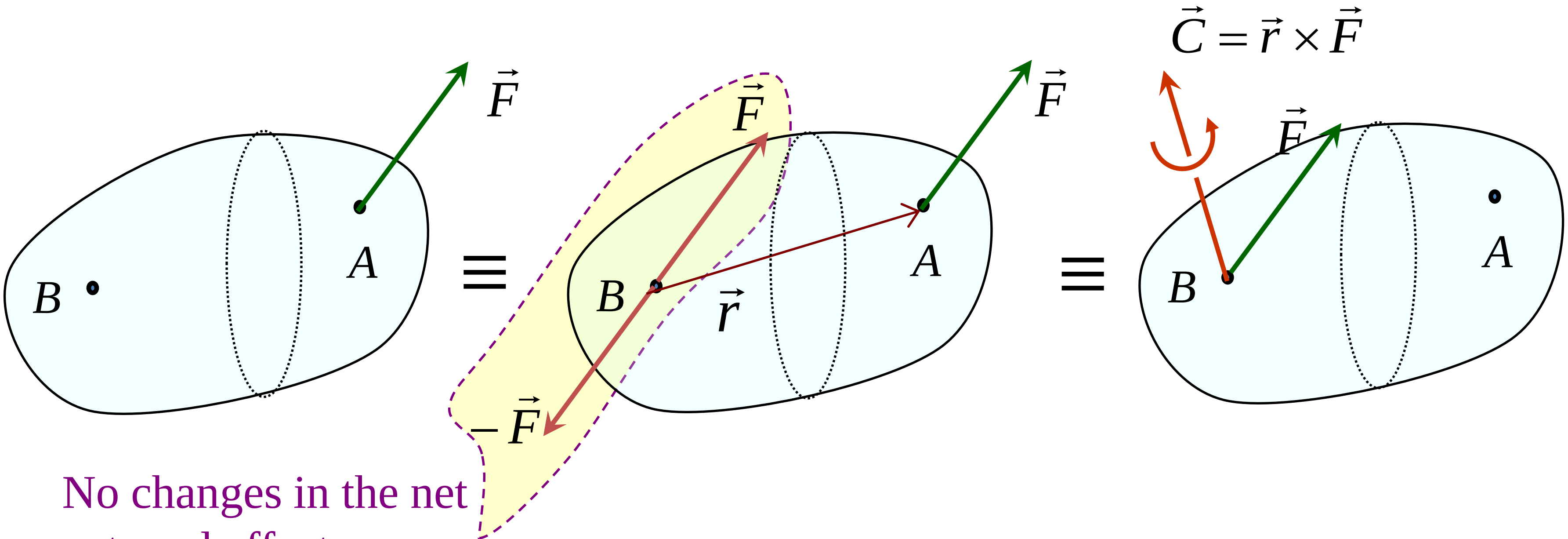
- Use and manipulate moment vectors
  - As couples are created from forces, why do we write down the couple vectors instead of forces in diagrams?
  - Given a couple of a point *P*, what do you know of the couple about a different point *Q*?
  - If we know moments about different points or axes, why can't we add components of moments as in vector summation?
  - Why can we simply add couple components together?

Resultant Definition

- The "force-couple systems" or "force systems" can be reduced to a single force and a single couple (together called resultant) that exert the same effects of
  - **Net force** ← Tendency to translate
  - **Net moment** ← Tendency to rotate

- Two force-couple systems are equivalent if their resultants are the same.

Force – Couple Systems



No changes in the net external effects

$\vec{C} = \vec{r} \times \vec{F}$: Couple of $\vec{F}$ about point B

from new location (point *B*)

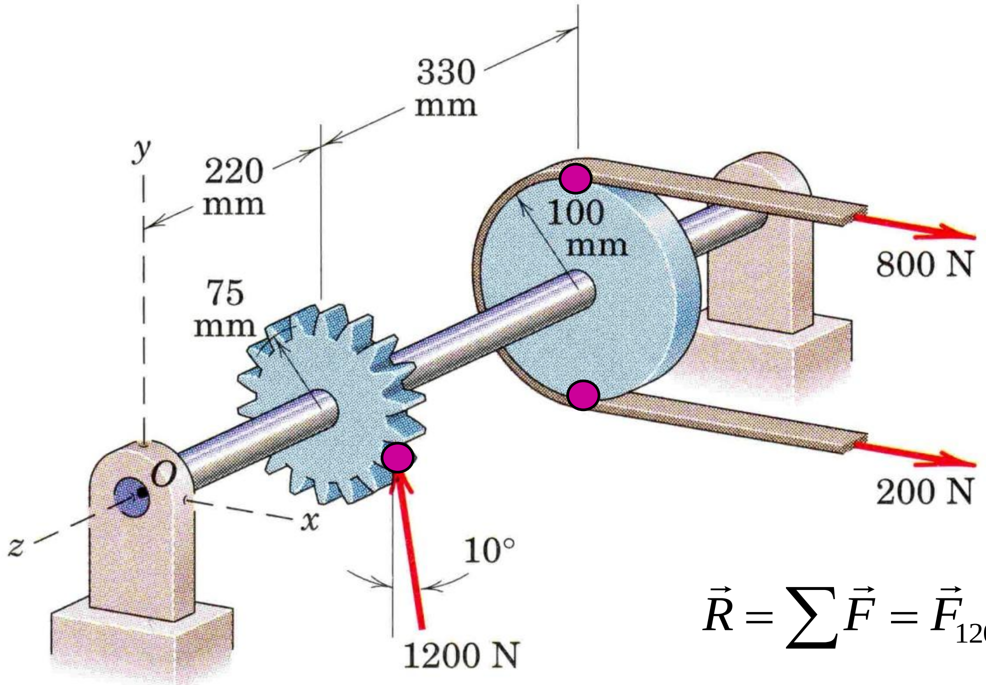
to any point on the line of action of $\vec{F}$

(which applied at the old point)

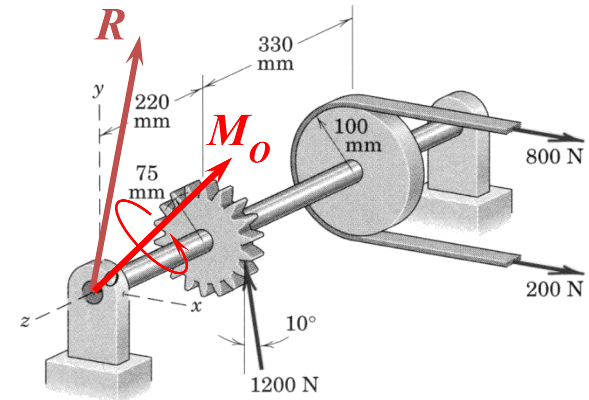
calculated the same way as "Moment of Point B by the force F at the old position"

2/139 The pulley and gear are subjected to the loads shown. For these forces, determine the equivalent force–couple system at point O.

Ans. $\mathbf{R} = 792\mathbf{i} + 1182\mathbf{j}$ N
$\mathbf{M}_O = 260\mathbf{i} - 504\mathbf{j} + 28.6\mathbf{k}$ N·m



Problem 2/139



Vector diagram

Move 3 forces to point O

$$\vec{F}_{1200} = 1200\left(-\sin 10^o \hat{i} + \cos 10^o \hat{j}\right)$$

$$\vec{F}_{800} = 800\hat{i} \qquad \vec{F}_{200} = 200\hat{i}$$

$$\vec{R} = \sum \vec{F} = \vec{F}_{1200} + \vec{F}_{800} + \vec{F}_{200} = 792\hat{i} + 1182\hat{j} \text{ N}$$ ***Ans***

$$\vec{M}_O = \sum \vec{M} = \vec{M}_{O,1200} + \vec{M}_{O,800} + \vec{M}_{O,200}$$

$$\vec{r}_{1200} = 0.075\hat{i} - 0.220\hat{k}$$

$$\vec{r}_{800} = 0.1\hat{j} - 0.55\hat{k}$$

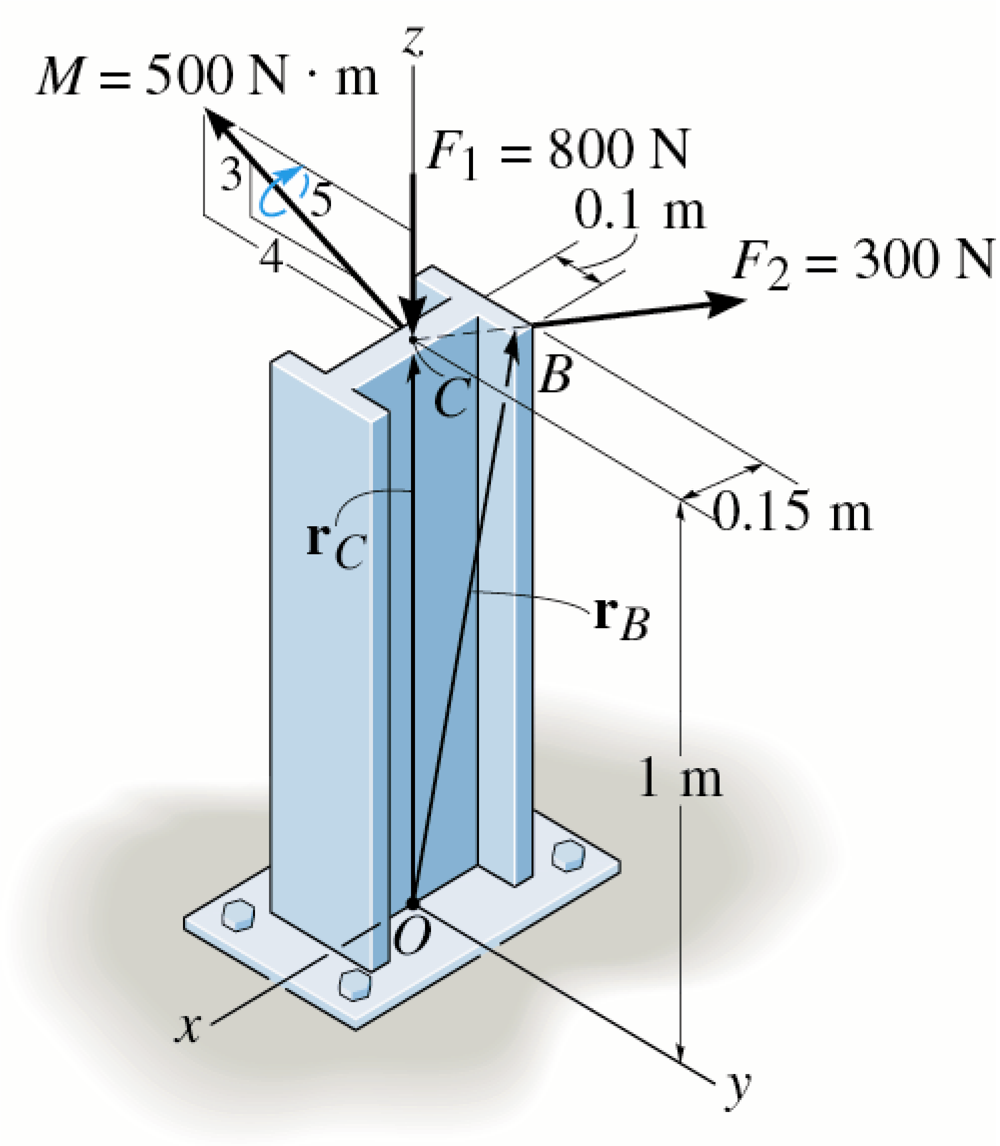
$$\vec{r}_{200} = -0.1\hat{j} - 0.55\hat{k}$$

$$= \vec{r}_{1200} \times \vec{F}_{1200} + \vec{r}_{800} \times \vec{F}_{800} + \vec{r}_{200} \times \vec{F}_{200}$$

$$= \left(260\hat{i} + 45.8\hat{j} + 88.6\hat{k}\right) + \left(-440\hat{j} - 80\hat{k}\right) + \left(-110\hat{j} + 20\hat{k}\right)$$

$$= 260\hat{i} - 504\hat{j} + 28.6\hat{k} \quad \text{N-m}$$ ***Ans***

Example Hibbeler Ex 4-15 #1



Replace the current system by an equivalent resultant force and couple moment acting at its base, point *O*.

$$\vec{F}_1 = -800\hat{k}\ \text{N}$$

$$\vec{F}_2 = 300\vec{u}_{CB} = 300(\vec{r}_{CB}/r_{CB})$$

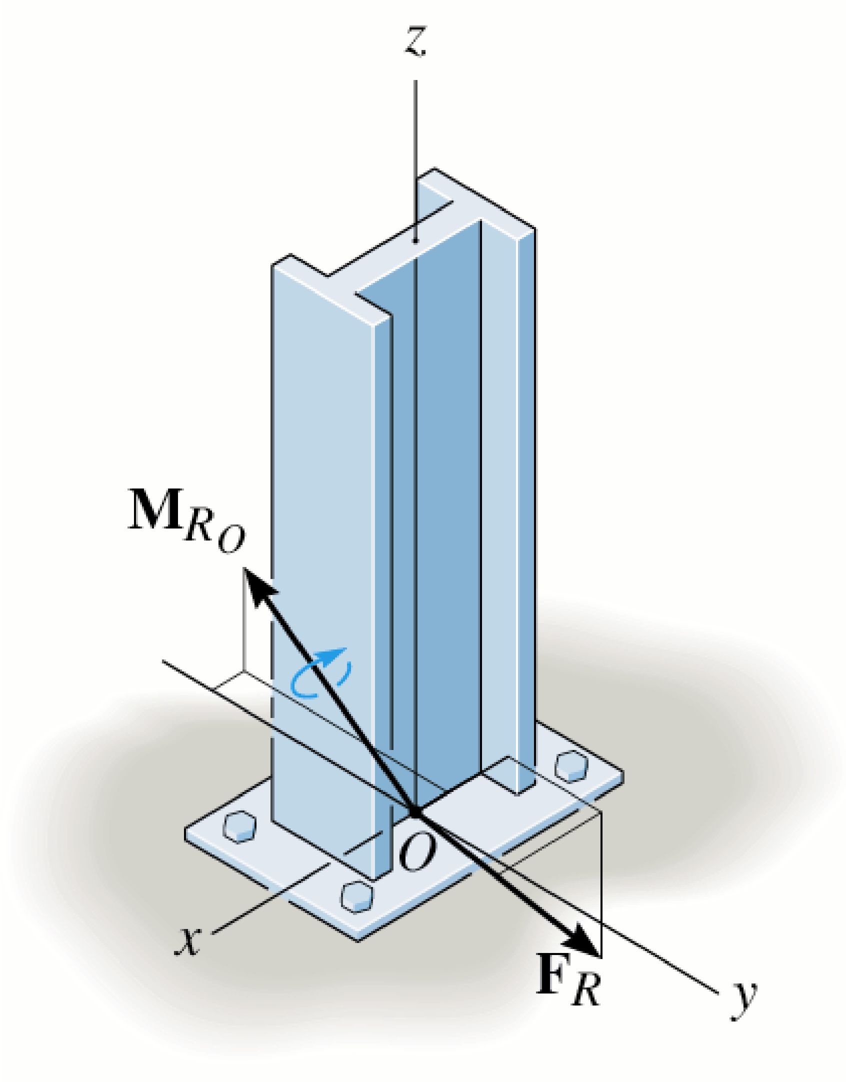
$$= 300\left[\frac{-0.15\hat{i} + 0.1\hat{j}}{\sqrt{(-0.15)^2 + (0.1)^2}}\right]$$

$$= (-249.62\hat{i} + 166.41\hat{j})\ \text{N}$$

$$\vec{M} = 500(-\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k})$$

$$= (-400\hat{j} + 300\hat{k})\ \text{N} \cdot \text{m}$$

Example Hibbeler Ex 4-15 #2



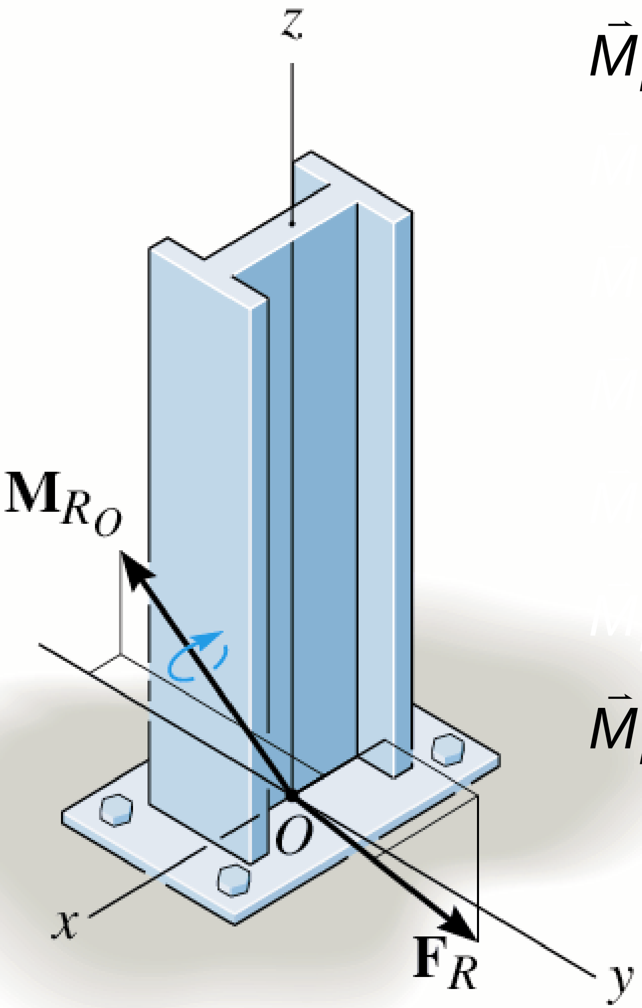
$$\left[\vec{F}_R = \sum \vec{F}\right]$$

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2$$

$$= (-249.62\hat{i} + 166.41\hat{j} - 800\hat{k})\ \text{N}$$

$$\vec{F}_R = (-250\hat{i} + 166\hat{j} - 800\hat{k})\ \text{N} \qquad \#$$

Example Hibbeler Ex 4-15 #3



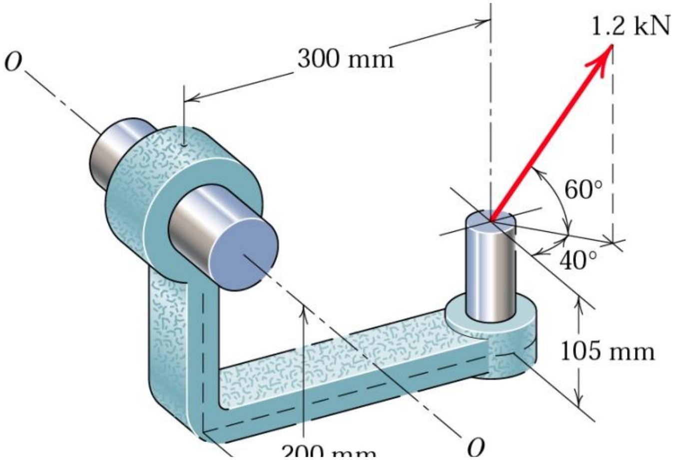
$$\left[\vec{M}_{R_O} = \sum \vec{M}\right]$$

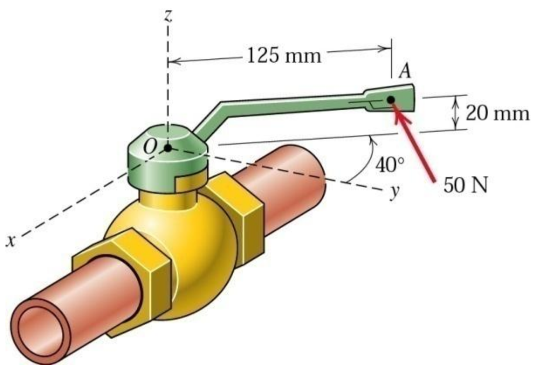
$$\begin{aligned}
\vec{M}_{R_O} &= \sum \vec{M}_C + \sum \vec{M}_O \\
&= \vec{M} + (\vec{r}_C \times \vec{F}_1) + (\vec{r}_B \times \vec{F}_2) \\
&= (-400\hat{j} + 300\hat{k}) + (1\hat{k}) \times (-800\hat{k}) + \\
&\quad (-0.15\hat{i} + 0.1\hat{j} + \hat{k}) \times (-249.62\hat{i} + 166.41\hat{j}) \\
&= (-400\hat{j} + 300\hat{k}) + (\vec{0}) + \\
&\quad (-166.41\hat{i} - 249.62\hat{j} + 0.0005\hat{k}) \\
\vec{M}_{R_O} &= (-166\hat{i} - 650\hat{j} + 300\hat{k}) \text{ N} \cdot \text{m} \qquad \#
\end{aligned}$$

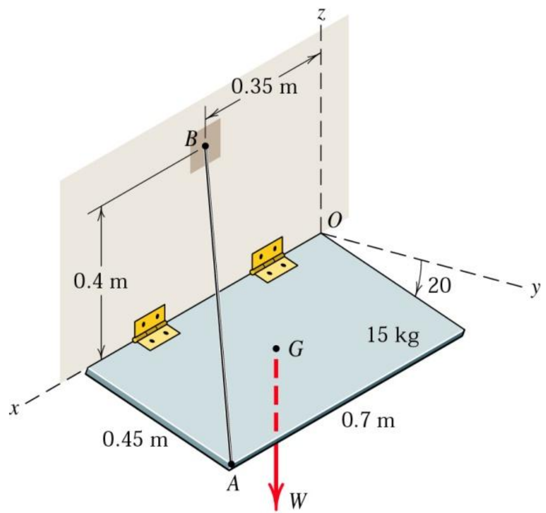
Recommended Problems

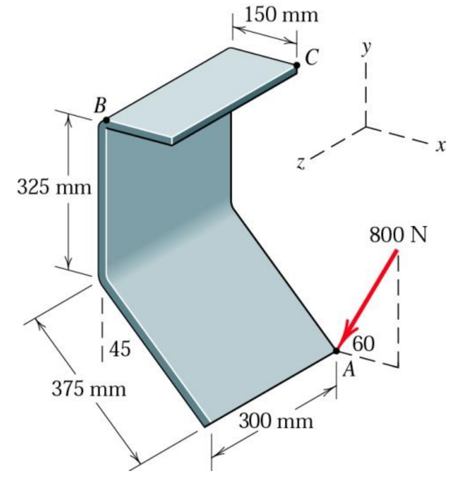
- 3D Moment and Couples:

  2/124 2/125   2/129 2/132

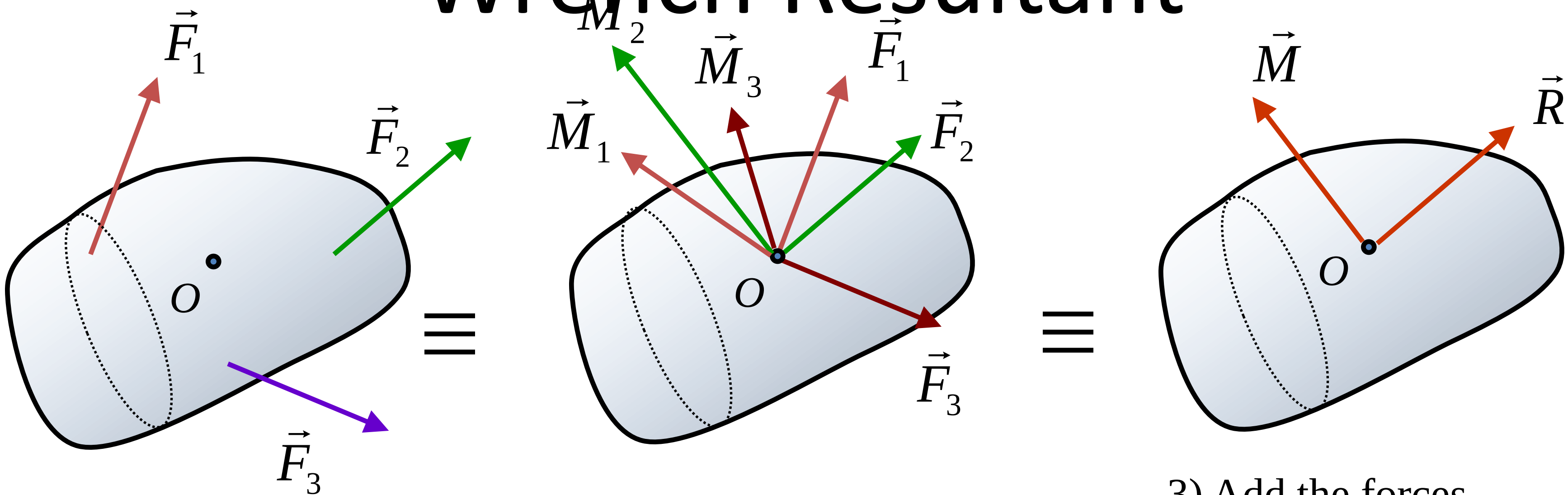








Wrench Resultant



$\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \ldots = \sum(\vec{r} \times \vec{F})$

1) Pick a point (easy to find moment arms)

2) Replace each force with a force at point O + a couple

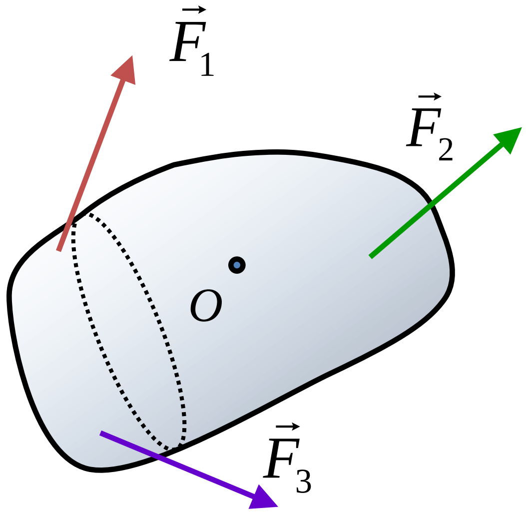
$$\vec{M}_1 = \vec{r}_1 \times \vec{F}_1$$

- $\vec{r}_1$ start from ____________

to______________

3) Add the forces vectorially to get the resultant force (since the forces are concurrent now) and add the couple vectorially to get the resultant couple

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = \sum \vec{F}$$



$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = \sum \vec{F}$$

$$\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \ldots = \sum (\vec{r} \times \vec{F})$$

$$R_x = \sum F_x \qquad R_y = \sum F_y \qquad R_z = \sum F_z$$

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2 + (\sum F_z)^2}$$

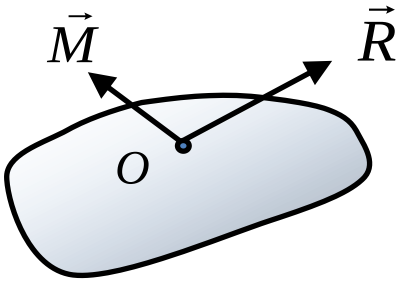
Vector

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = (r_y F_z - r_z F_y)\hat{i} + (r_z F_x - r_x F_z)\hat{j} + (r_x F_y - r_y F_x)\hat{k}$$

Scalar
(2D * 3Plane)

$$M_x = \sum (\vec{r} \times \vec{F})_x \qquad M_y = \sum (\vec{r} \times \vec{F})_y \qquad M_z = \sum (\vec{r} \times \vec{F})_z$$

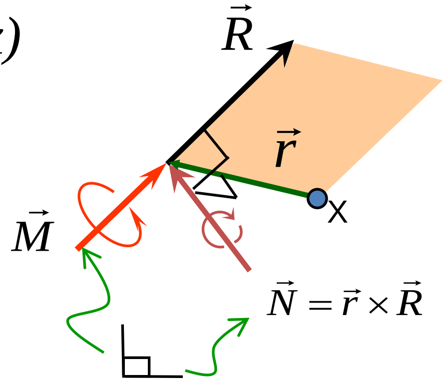
$$M = \sqrt{M_x^2 + M_y^2 + M_y^2}$$



- The choice of point O is arbitrary;

the resultant couple will not be the same for each point O selected (in general), but the resultant force will be the same.

Ex)



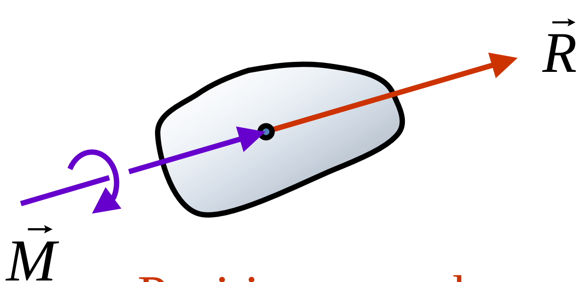
- The resultant couple cannot be cancelled by moving the resultant force (in general).

M which // R, cannot be cancelled

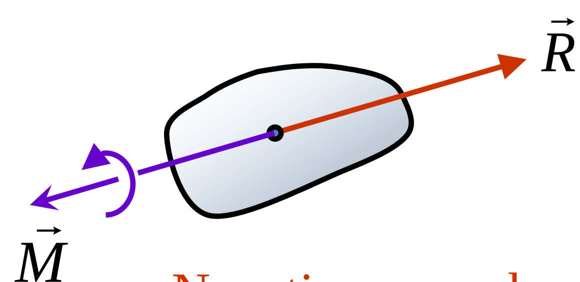
M which |_ R, can be cancelled.

Wrench Resultant (not very useful)

- All force systems can be represented with a wrench resultant as shown in the figures



Positive wrench



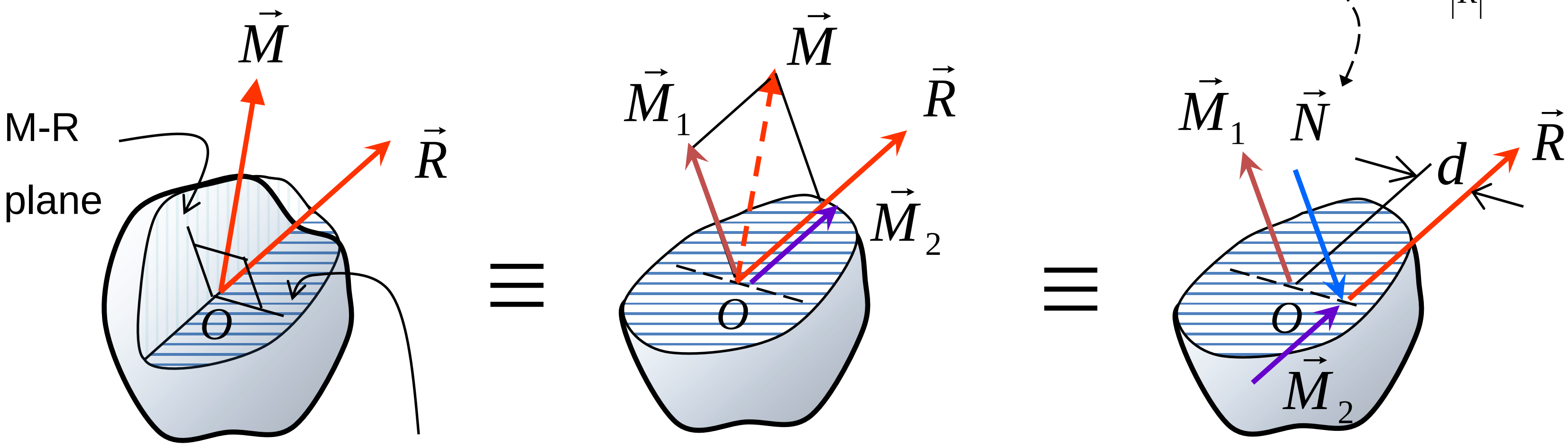
Negative wrench

Positive if right-hand rule

How to find Wrench Resultant

for $\vec{N} = -\vec{M}_1$

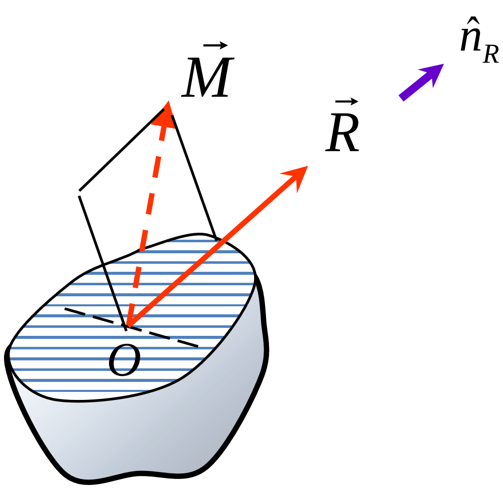
$$d = \frac{|\vec{M}_1|}{|\vec{R}|} \quad \text{(move to right)}$$



Plane Containing $\vec{R}$

and $\perp$ with M-R plane

Vector approach see ex. 2/16



How to find $\vec{M}_1, \vec{M}_2$

(knowing $\vec{M}, \vec{R}$)

$$\vec{M}_2 = \{ \vec{M} \cdot \hat{n}_R \} \hat{n}_R$$

$$\vec{M}_1 = \vec{M} - \vec{M}_2$$

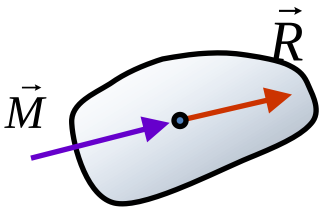
The simplest form of force-couple system

3D

any forces + couples system

⇨ single-force + single couple
(which // with each other)

wrench resultant

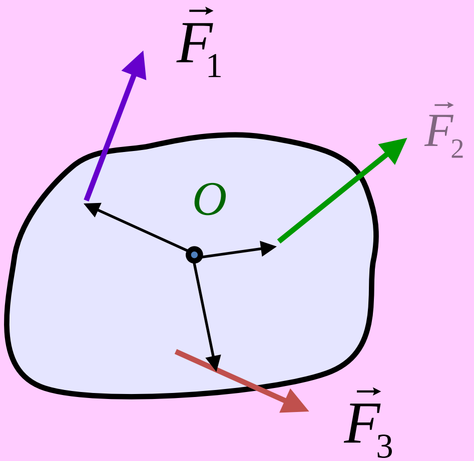


2D

any forces + couples system

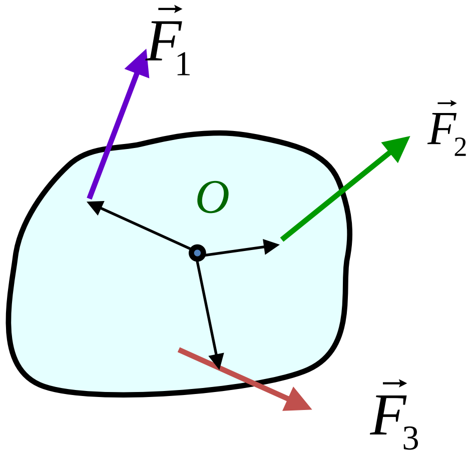
⇨ single-force system (no-couple)

OR single-couple system

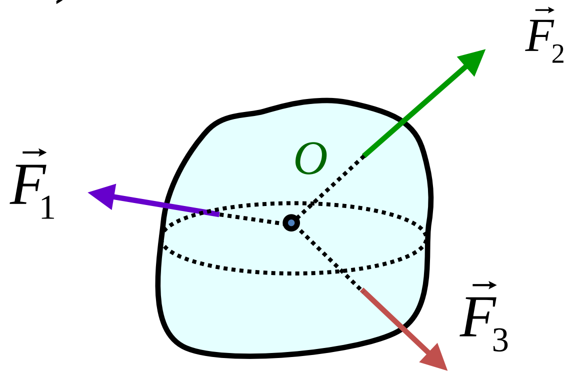


Why 2D is different from 3D?

Special cases: Wrench Resultant



1) Coplanar: 2D (Article 2/6)



2) Concurrent force:

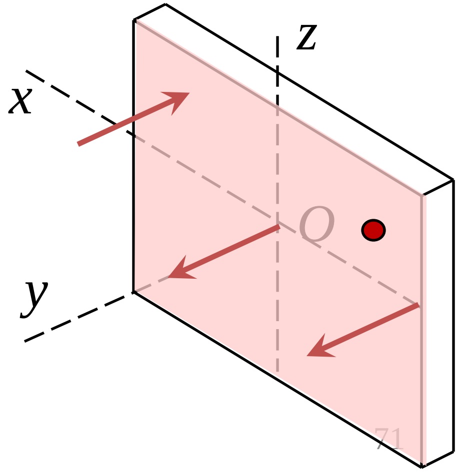
the resultant will pass through the point of concurrency.

No resultant moment at concurrent point. Pick the point of concurrency!

3) Parallel forces (not in same plane):

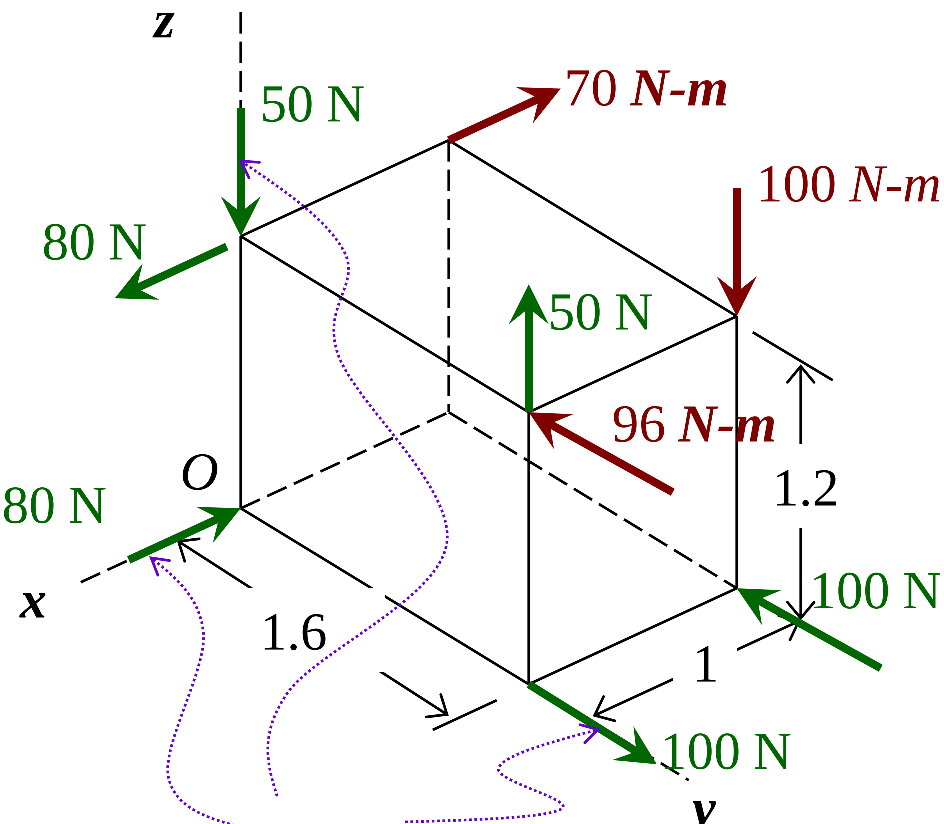
single-force system (no-couple)

OR single-couple system



Sample problem 2/13

Find the resultant



pass thru O: no need to calculate couple

Move all force to point O

$$\vec{R} = (80-80)\hat{i} + (100-100)\hat{j} + (50-50)\hat{k} = \vec{0} \quad \text{N}$$

$$\vec{M} = +80(1.2)\hat{j} + 50(1.6)\hat{i} + 100(1)\hat{k} - 70\hat{i} - 96\hat{j} - 100\hat{k} = 10\hat{i} \quad \text{N-m}$$

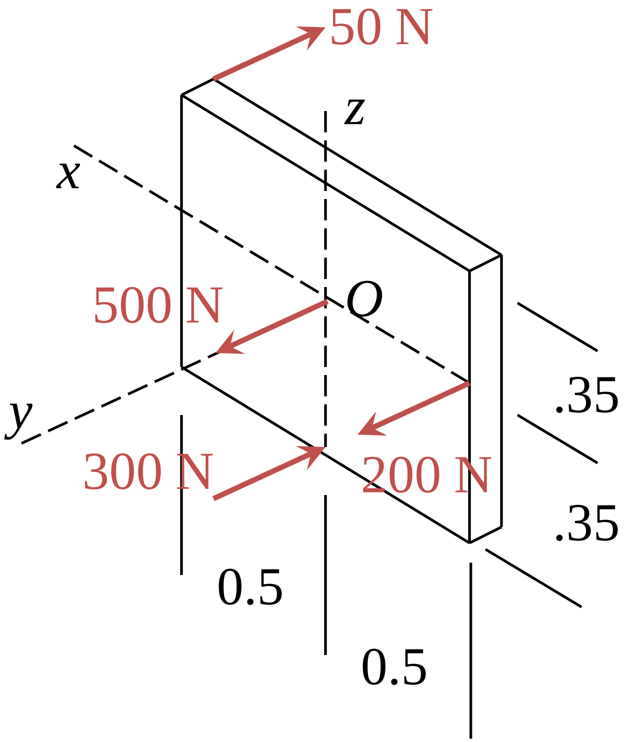
Ans

$$\vec{R} = \vec{0}$$

$$\vec{M} = +80(1.2)\hat{j} + 100(1)\hat{k} + 50(1.6)\hat{i} - 70\hat{i} - 96\hat{j} - 100\hat{k}$$

$$= 10\hat{i} \quad \text{N-m}$$

Find the resultant



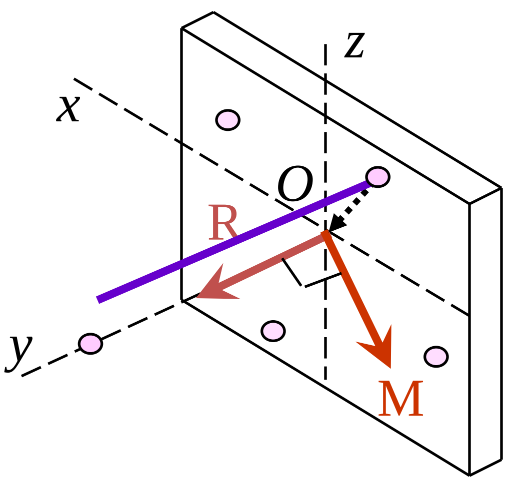
Move all force to point O

$$\vec{R} = (-50 + 500 + 200 - 300)\,\hat{j} = 350\,\hat{j} \quad \text{N}$$

$$\vec{M}_O = +50(0.35)\hat{i} - 50(0.5)\hat{k}$$
$$-200(0.35)\hat{k}$$
$$-300(0.35)\hat{i}$$
$$= -87.5\hat{i} - 125\hat{k} \quad \text{N-m}$$

R |_ M

Moving R can erase M completely



New point: (x,y,z)

Couple cancelled: $\vec{M}_O + (-x\hat{i} - y\hat{j} - z\hat{k}) \times \vec{R} = \vec{0}$

Equivalent System (at O): $(x\hat{i} + y\hat{j} + z\hat{k}) \times \vec{R} = \vec{M}_O$

$$(350x)\hat{k} - (350z)\hat{i} = -87.5\hat{i} - 125\hat{k}$$

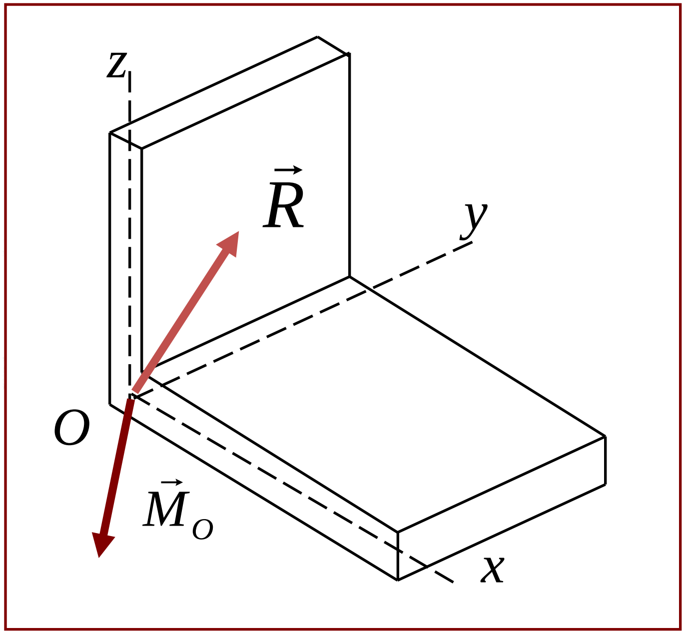
$$x = \frac{-125}{350} = -0.357 \quad z = \frac{-87.5}{-350} = 0.250$$

y : any value

Which quadrant?

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.

Solution 0 (Wrong)



Move all force to point O

$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$

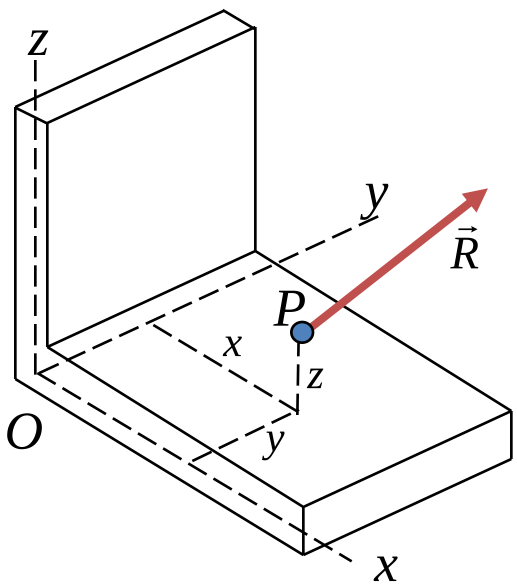
$$\vec{M}_O = 0.06\hat{k} \times 40\hat{j} + (0.1\hat{i} + 0.08\hat{j}) \times 40\hat{k}$$

$$= -2.4\hat{i} - 4\hat{j} + 3.2\hat{i} = 0.8\hat{i} - 4\hat{j}$$

Move ***R*** to point P (x,y,z), to cancel the couple

Couple cancelled: $\vec{M}_O + (-x\hat{i} - y\hat{j} - z\hat{k}) \times \vec{R} = \vec{0}$

Equivalent System (at O): $\vec{M}_O = (x\hat{i} + y\hat{j} + z\hat{k}) \times \vec{R}$



$$40y - 40z = 0.8$$
$$20z - 40x = -4$$
$$40x - 20y = 0$$

$\Rightarrow$

$$y - z = 0.02$$
$$z - 2x = -0.2$$
$$2x - y = 0$$

$\Rightarrow$

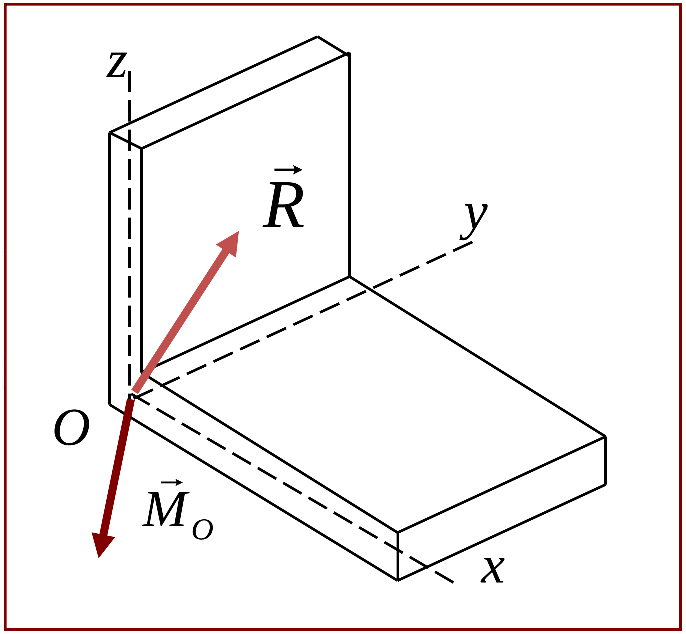
$$2x - z = 0.02$$
$$z - 2x = -0.2$$

unable to solve!!

Generally in 3D, we can not change force-couple system to single-force system.

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.

Solution 1: Direct Method



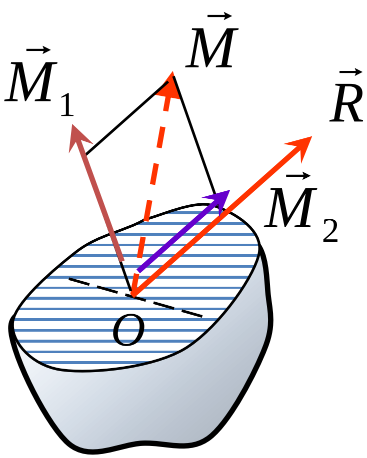
Move all force to point O

$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$

$$\vec{M}_O = 0.06\hat{k} \times 40\hat{j} + (0.1\hat{i} + 0.08\hat{j}) \times 40\hat{k}$$

$$= -2.4\hat{i} - 4\hat{j} + 3.2\hat{i} = 0.8\hat{i} - 4\hat{j}$$

$$\hat{n}_{\vec{R}} = \frac{1}{\sqrt{20^2 + 40^2 + 40^2}}\left(20\hat{i} + 40\hat{j} + 40\hat{k}\right)$$



$$\vec{M}_{O,//} = (\vec{M}_O \cdot \hat{n}_{\vec{R}})\ \hat{n}_{\vec{R}}$$

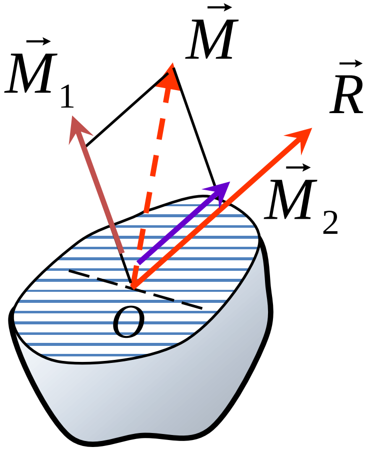
negative wrench

$$= (\frac{-144}{60}) \quad \frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k})$$

$$= -(0.8\hat{i} + 1.6\hat{j} + 1.6\hat{k})$$

$$\vec{M}_{O,\perp} = \vec{M}_O - (\vec{M}_O \cdot \hat{n}_{\vec{R}})\hat{n}_{\vec{R}} = 1.6\hat{i} - 2.4\hat{j} + 1.6\hat{k}$$

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.



$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N} \qquad \vec{M}_O = 0.8\hat{i} - 4\hat{j}$$

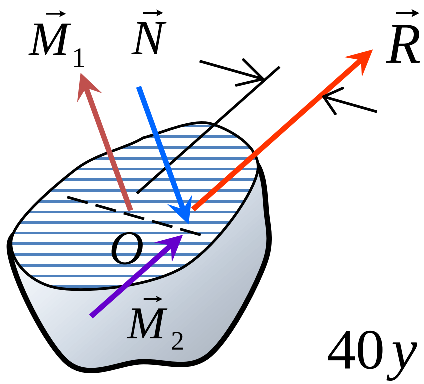
$$\vec{M}_{O,//} = (\vec{M}_O \cdot \hat{n}_{\vec{R}})\hat{n}_{\vec{R}} = -(0.8\hat{i} + 1.6\hat{j} + 1.6\hat{k}) \quad \text{N-m}$$

$$\vec{M}_{O,\perp} = \vec{M}_O - (\vec{M}_O \cdot \hat{n}_{\vec{R}})\hat{n}_{\vec{R}} = 1.6\hat{i} - 2.4\hat{j} + 1.6\hat{k}$$

new point P: (x,y,z) old point O: (0,0,0)

$$\boxed{\vec{M}_{O,\perp} + \vec{N} = \vec{0}} \qquad -(x\hat{i} + y\hat{j} + z\hat{k}) \times \vec{R} + \vec{M}_{o,\perp} = \vec{0}$$

$\equiv$



$$(x\hat{i} + y\hat{j} + z\hat{k}) \times (20\hat{i} + 40\hat{j} + 40\hat{k})$$
$$= 1.6\hat{i} - 2.4\hat{j} + 1.6\hat{k}$$

line of action

$$40y - 40z = 1.6$$
$$20z - 40x = -2.4$$
$$40x - 20y = 1.6$$

$$y - z = 0.4$$
$$2x - z = 0.12$$

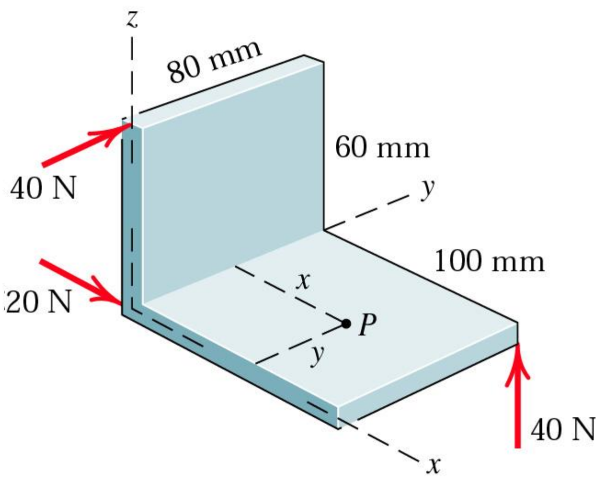
$$y = z + 0.4 = 2x - 0.8$$

$$z = 0: \quad x = 0.6 \quad y = 0.4$$

Ans

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.

Solution 2: Equivalent System



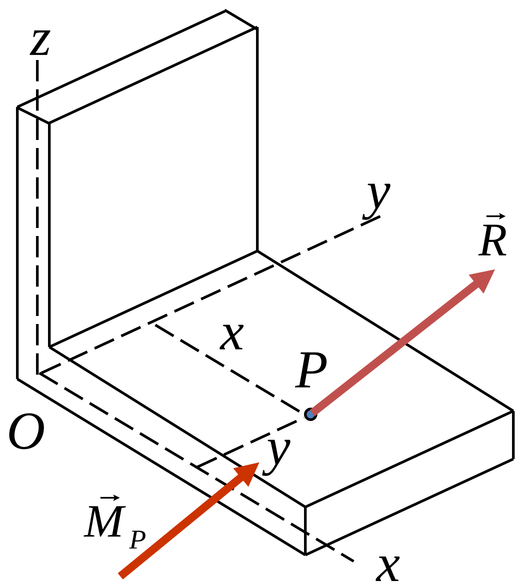
Assume (x,y,0) is the point where wrench passes.

$$\vec{M}_{O,Sys1} = 0.06\hat{k} \times 40\hat{j} + (0.1\hat{i} + 0.08\hat{j}) \times 40\hat{k}$$

$$= -2.4\hat{i} - 4\hat{j} + 3.2\hat{i} = 0.8\hat{i} - 4\hat{j}$$

$$\boxed{\vec{M}_{O,Sys1} = \vec{M}_{O,Sys2}} \qquad \boxed{\vec{R}_{Sys1} = \vec{R}_{Sys2}}$$

$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$



$$\vec{M}_{O,Sys2} = (x\hat{i} + y\hat{j}) \times \vec{R} + \vec{M}_P$$

$$= (40y)\hat{i} - (40x)\hat{j} + (40x - 20y)\hat{k} + \vec{M}_P$$

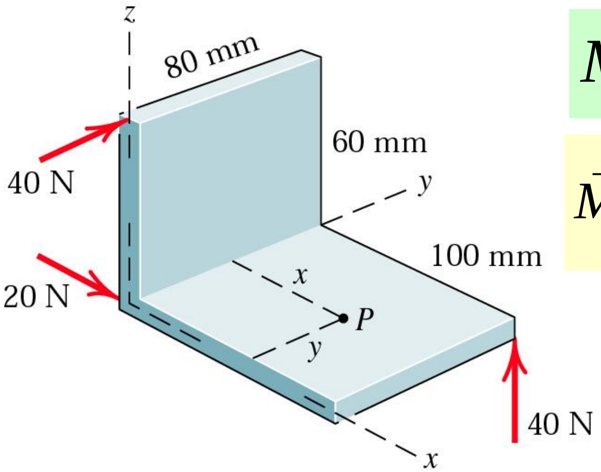
Parallel Condition

$$\vec{M}_P = M\ \hat{n}_{\vec{R}} = M\frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k})$$

M (+ or – is ok)

$$\vec{M}_{O,Sys2} = (40y + \frac{M}{3})\hat{i} + (-40x + \frac{2M}{3})\hat{j} + (40x - 20y + \frac{2M}{3})\hat{k}$$

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.



$$\vec{M}_{O,Sys1} = -2.4\hat{i} - 4\hat{j} + 3.2\hat{i} = 0.8\hat{i} - 4\hat{j}$$

$$\vec{M}_{O,Sys2} = (40y + \frac{M}{3})\hat{i} + (-40x + \frac{2M}{3})\hat{j} + (40x - 20y + \frac{2M}{3})\hat{k}$$

$$\boxed{\vec{M}_{O,Sys1} = \vec{M}_{O,Sys2}}$$

$$40y + \frac{M}{3} = 0.8$$

$$-40x + \frac{2M}{3} = -4$$

$$40x - 20y + \frac{2M}{3} = 0$$

$$\left.\begin{aligned} x &= 0.06 \\ y &= 0.04 \end{aligned}\right\}$$

The coordinate in x-y plane, where wrench resultant passes

$$M = -2.4$$

Magnitude: 2.4 N-m

Direction: opposite with ***R***

(negative wrench)

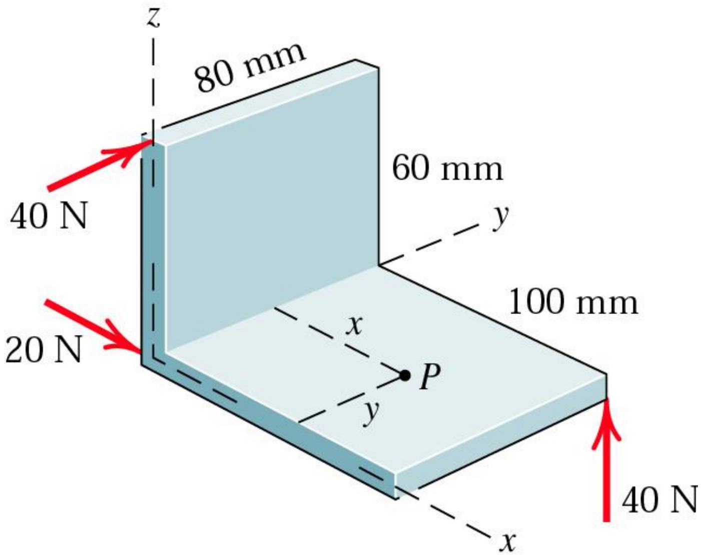
$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$ (negative wrench)

$$\vec{M} = -2.4\frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k}) \quad \text{N-m}$$

$$P : (0.06, 0.04) \ \text{m}$$ ***Ans***

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.

$\boxed{\text{Solution 3: wrench condition}}$



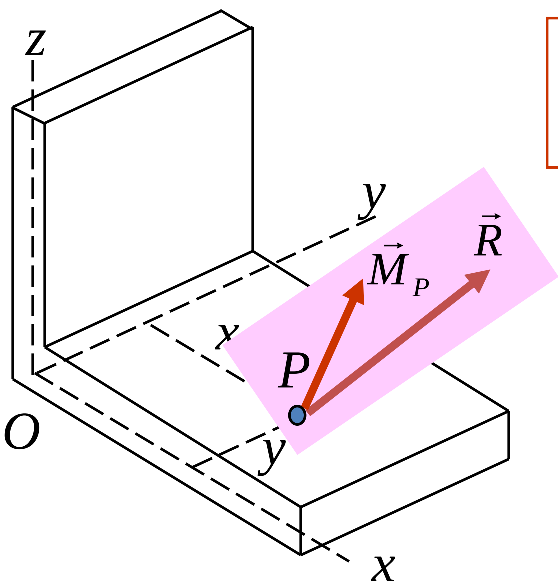
Move forces to P (x,y,0)

$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$

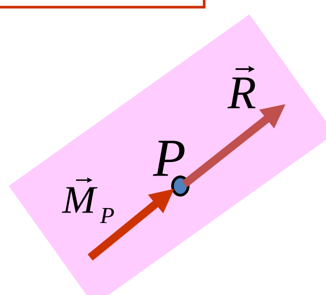
$$|\vec{R}| = \sqrt{20^2 + 40^2 + 40^2} = 60$$

$$\vec{M}_P = (-x\hat{i} - y\hat{j}) \times 20\hat{i}$$
$$+(-x\hat{i} - y\hat{j} + 0.06\hat{k}) \times 40\hat{j}$$
$$+((0.1 - x)\hat{i} + (0.08 - y)\hat{j}) \times 40\hat{k}$$

$$= (0.8 - 40y)\hat{i} + (40x - 4)\hat{j} + (20y - 40x)\hat{k} \quad \text{N-m}$$



$\boxed{\text{wrench condition}}$



$$\boxed{\hat{n}_{\vec{R}} = \pm \hat{n}_{\vec{M}_P}}$$

$$\hat{n}_{\vec{R}} = \frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k})$$

$$\hat{n}_{\vec{M}_P} = \frac{1}{|\vec{M}_P|}\left\{(0.8 - 40y)\hat{i} + (40x - 4)\hat{j} - (20y - 40x)\hat{k}\right\}$$

$$\sqrt{(0.8 - 40y)^2 + (40x - 4)^2 - (20y - 40x)^2}$$

Take it as
the other unknown

Find the wrench resultant, give coordinates on x-y plane that the wrench resultant acts.

$$\hat{n}_{\vec{R}} = \frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k})$$

$$\boxed{\hat{n}_{\vec{R}} = \pm\hat{n}_{\vec{M}_P}}$$

$$\hat{n}_{\vec{M}_P} = \frac{1}{|\vec{M}_P|}(0.8 - 40y)\hat{i} + (40x - 4)\hat{j} + (20y - 40x)\hat{k}$$

$$\frac{20}{60}\hat{i} + \frac{40}{60}\hat{j} + \frac{40}{60}\hat{k} = \pm\frac{1}{|\vec{M}_P|}(0.8 - 40y)\hat{i} + (40x - 4)\hat{j} + (20y - 40x)\hat{k}$$

M (+ or – is ok)

$$\frac{M}{3} = 0.8 - 40y$$

$$\frac{2M}{3} = 40x - 4$$

$$\frac{2M}{3} = 20y - 40x$$

$$x = 0.06$$

$$y = 0.04$$

The coordinate in x-y plane, where wrench resultant passes

$$M = -2.4$$

Magnitude: 2.4 N-m

$$\vec{R} = 20\hat{i} + 40\hat{j} + 40\hat{k} \quad \text{N}$$ (negative wrench)

$$\vec{M} = -2.4\frac{1}{60}(20\hat{i} + 40\hat{j} + 40\hat{k}) \quad \text{N-m}$$

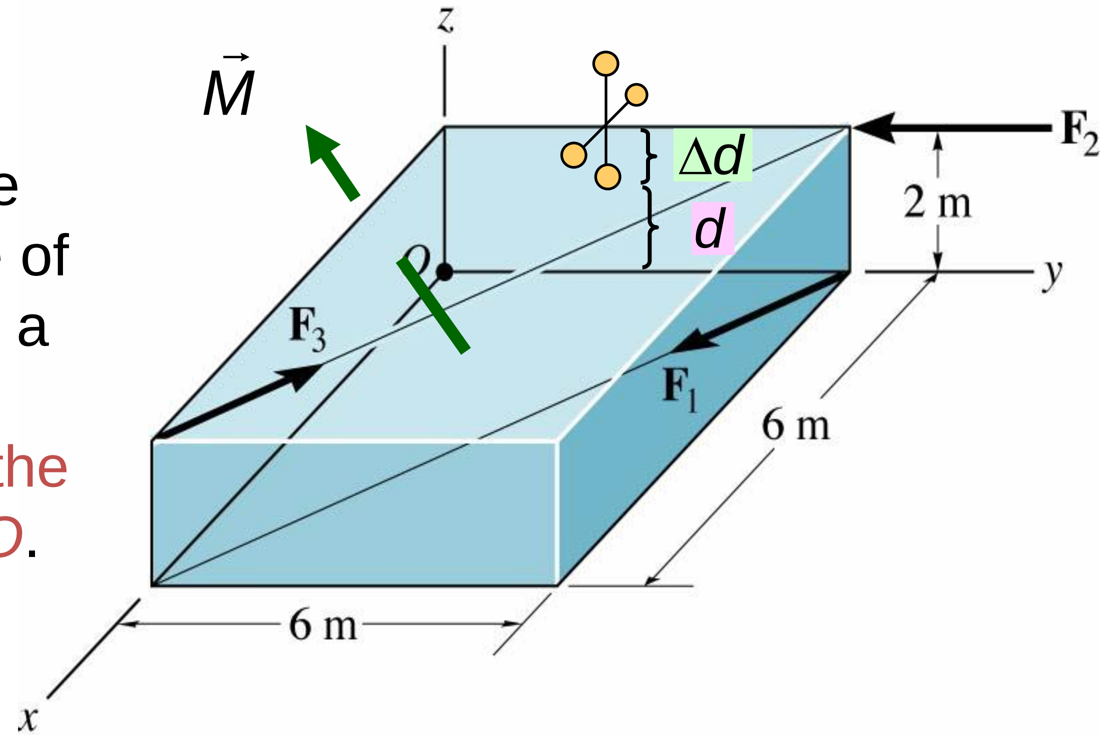
$$P: (0.06, 0.04) \ \text{m}$$

Ans

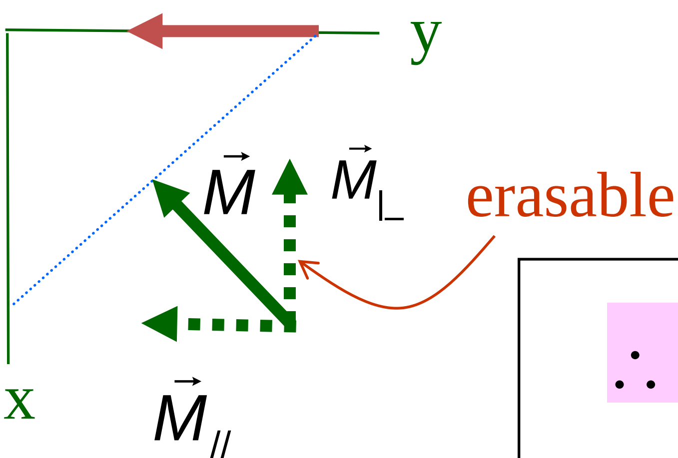
Direction: opposite with ***R*** (negative wrench)

Hibbeler Ex 4-136

The three forces acting on the block each have a magnitude of 10 N. Replace this system by a wrench and specify the point where the wrench intersects the z axis, measured from point *O*.



$\vec{R} = \vec{F}_3$



$$\Delta d = \frac{|\vec{M}_{\perp}|}{|\vec{R}|} = \frac{(10 \times 2)(\cos 45^o)}{10} = \sqrt{2} \quad (\text{dir:} \downarrow)$$

$$\therefore \quad d = 2 - \sqrt{2} \text{ m}$$

$$\vec{R} = -10\hat{j} \text{ N}$$

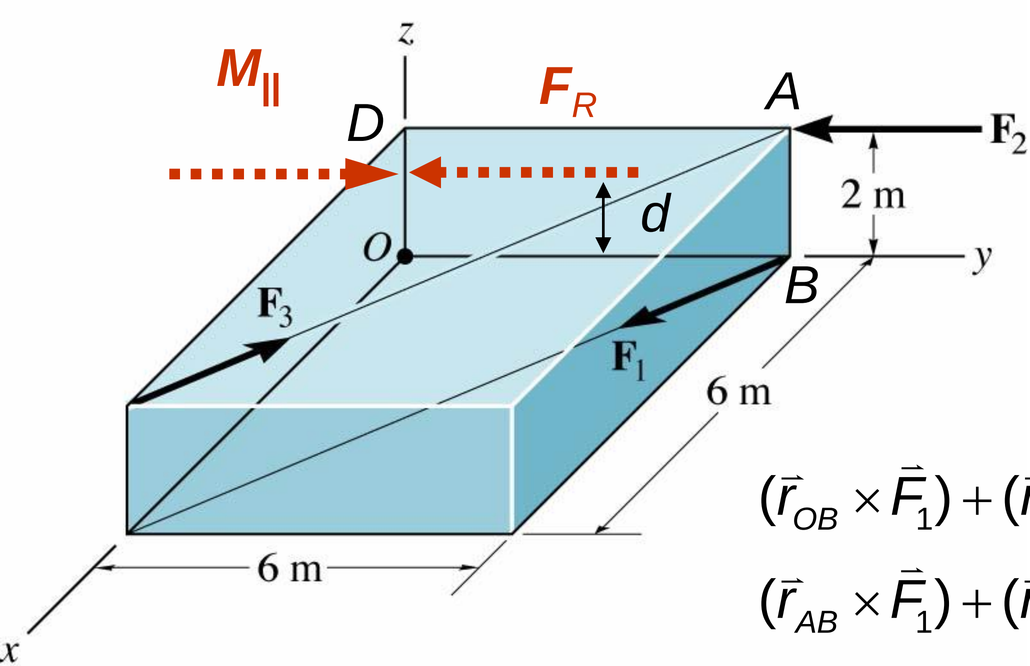
Positive wrench

$$\vec{M}_{//} = (10 \times 2 \cos 45)(-\hat{j}) = -10\sqrt{2}\hat{j} \text{ N-m}$$ *Ans*

Hibbeler Ex 4-136

$$\vec{F}_R = -10\hat{j}\ \text{N}$$

$$\vec{M}_{\|} = M_{\|}\hat{j}\ \text{N}\cdot\text{m}$$



$$\sum \vec{M}_{O,Sys1} = \sum \vec{M}_{O,Sys2}$$

$$(\vec{r}_{OB} \times \vec{F}_1) + (\vec{r}_{OA} \times \vec{F}_2) + (\vec{r}_{OB} \times \vec{F}_3) = (\vec{r}_{OD} \times \vec{F}_R) + M_{\|}\hat{j}$$

$$(\vec{r}_{AB} \times \vec{F}_1) + (\vec{r}_{OA} \times \vec{F}_2) = (\vec{r}_{OD} \times \vec{F}_R) + M_{\|}\hat{j}$$

$$(-2\hat{k}) \times 10(\cos 45°\hat{i} - \sin 45°\hat{j}) + (6\hat{j} + 2\hat{k}) \times (-10\hat{j})$$

$$= (d\hat{k}) \times (-10\hat{j}) + M_{\|}\hat{j}$$

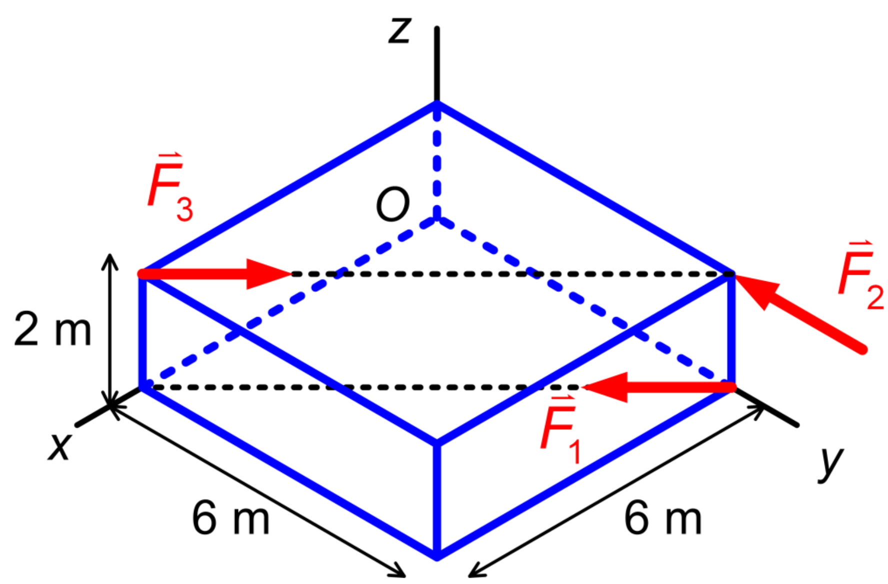
$$-10\sqrt{2}\hat{i} - 10\sqrt{2}\hat{j} + 20\hat{i} = 10d\hat{i} + M_{\|}\hat{j}$$

y-direction: $-10\sqrt{2} = M_{\|} \quad \rightarrow \quad M_{\|} = -10\sqrt{2} = -14.142\ \text{N}\cdot\text{m}$

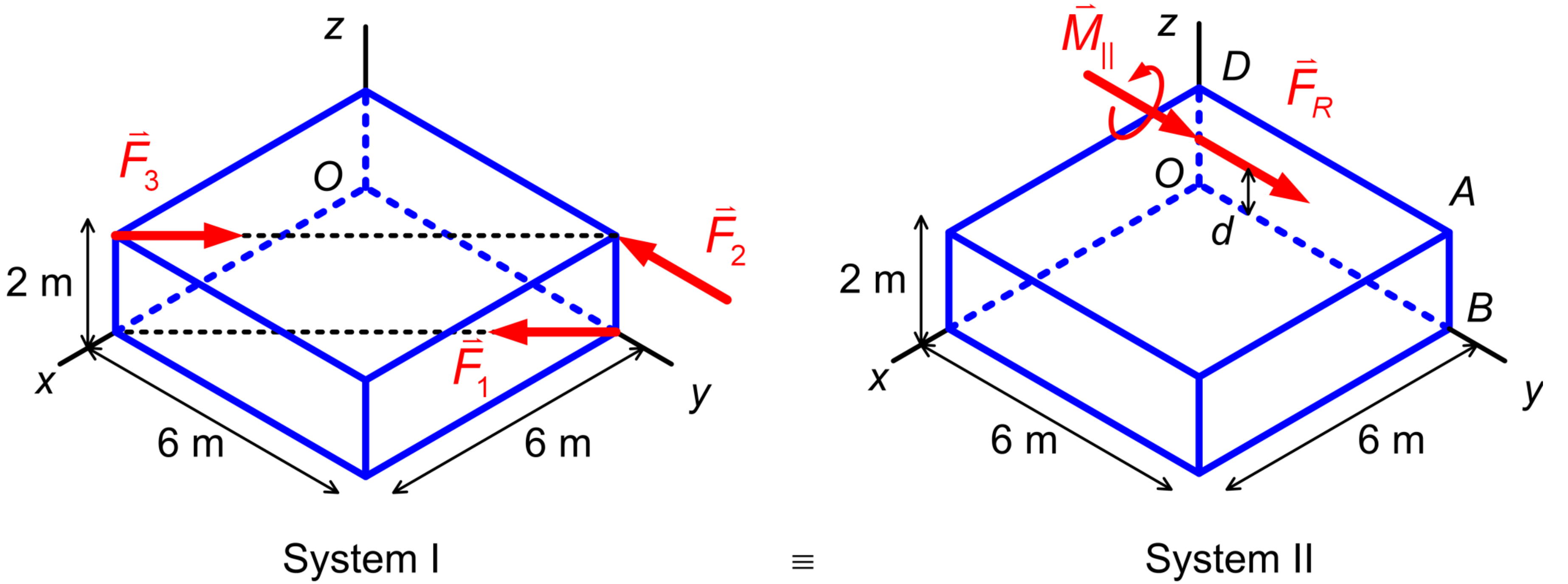
x-direction: $-10\sqrt{2} + 20 = 10d \quad \rightarrow \quad d = 0.58579\ \text{m}$

Example Hibbeler Ex 4-136 #1

The three forces acting on the block each have a magnitude of 10 N. Replace this system by a wrench and specify the point where the wrench intersects the z axis, measured from point *O*.



Example Hibbeler Ex 4-136 #2



$$\left[\sum \vec{F}_{\text{system I}} = \sum \vec{F}_{\text{system II}}\right]$$

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{F}_R$$

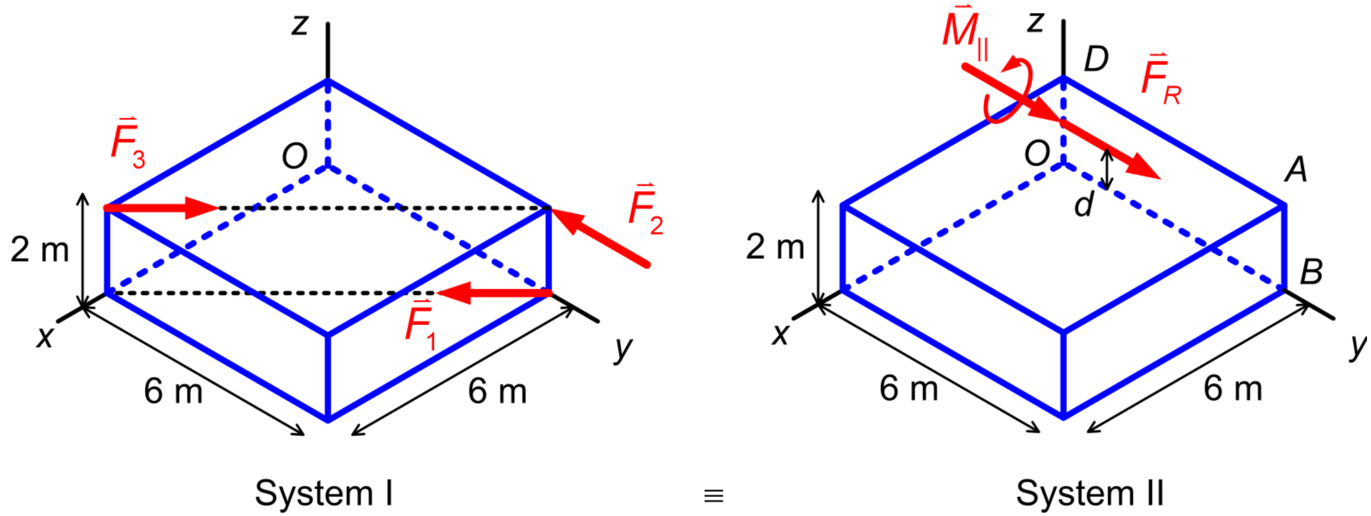
$$(10\ \text{N})(\cos 45°\hat{i} - \sin 45°\hat{j}) - (10\hat{j}\ \text{N}) + (10\ \text{N})(-\cos 45°\hat{i} + \sin 45°\hat{j}) = \vec{F}_R$$

$$\vec{F}_R = -10\hat{j}\ \text{N}$$

$$\vec{F}_R = -10\hat{j}\ \text{N}$$

Example Hibbeler Ex 4-136 #3

$$\vec{M}_{||} = M_{||}\hat{j}\ \text{N}\cdot\text{m}$$



$$\left[\sum \vec{M}_{\text{O,system I}} = \sum \vec{M}_{\text{O,system II}}\right]$$

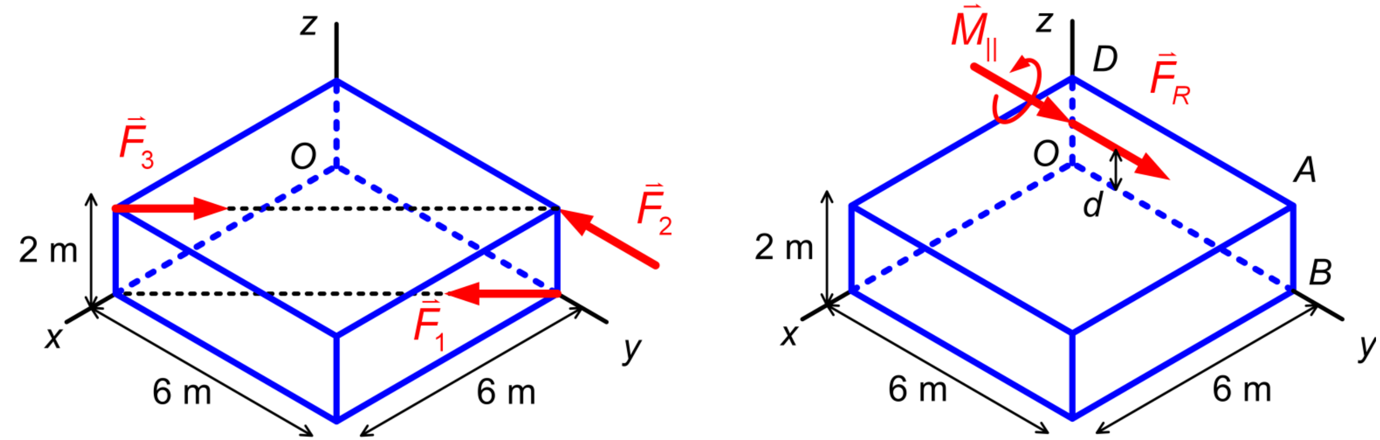
$$(\vec{r}_{OB} \times \vec{F}_1) + (\vec{r}_{OA} \times \vec{F}_2) + (\vec{r}_{OA} \times \vec{F}_3) = (\vec{r}_{OD} \times \vec{F}_R) + M_{||}\hat{j}$$

$$(\vec{r}_{AB} \times \vec{F}_1) + (\vec{r}_{OA} \times \vec{F}_2) = (\vec{r}_{OD} \times \vec{F}_R) + M_{||}\hat{j}$$

$$(-2\hat{k}\ \text{m}) \times (10\ \text{N})(\cos 45^\circ \hat{i} - \sin 45^\circ \hat{j}) + (6\hat{j} + 2\hat{k})\ \text{m} \times (-10\hat{j}\ \text{N}) =$$

$$(d\hat{k}) \times (-10\hat{j}\ \text{N}) + (M_{||}\hat{j}\ \text{N}\cdot\text{m})$$

Example **Hibbeler Ex 4-136 #4**



System I $\equiv$ System II

$$(-10\sqrt{2}\,\hat{i} - 10\sqrt{2}\,\hat{j} + 20\hat{i}) \text{ N}\cdot\text{m} = (10 \text{ N})d\hat{i} + M_{||}\hat{j}$$

y-direction: $\quad (-10\sqrt{2}) \text{ N}\cdot\text{m} = M_{||}$

$$M_{||} = -10\sqrt{2} \text{ N}\cdot\text{m} = -14.142 \text{ N}\cdot\text{m}$$

x-direction: $\quad (-10\sqrt{2} + 20) \text{ N}\cdot\text{m} = (10 \text{ N})d$

$$d = 0.58579 \text{ m}$$

Wrench:

$$\vec{F}_R = -10\hat{j} \text{ N}$$

$$\vec{M}_{||} = -14.1\hat{j} \text{ N}\cdot\text{m}$$

$$d = 0.586 \text{ m}$$ Ans

- **สรุปแนวทางการสอนที่จะเน้น**
- หน่วยให้ใส่ตลอด โดยในบรรทัดแรกให้ใส่ตามตัวแปรที่ยกมาตั้ง ส่วนบรรทัดต่อๆ ไป รวมเป็นหน่วยเดียวไว้ท้ายบรรทัดได้
- - สัญลักษณ์ สำหรับ moment/couple ให้ใช้ลูกศรที่มีวงบอกการหมุน ไม่ใช้หัวลูกศรคู่ และให้กำหนดแกน xy พร้อมทิศทวนเข็มเป็นบวกในการวาดรูปแกน
- - การนิยาม ให้ระวังให้มีการนิยามตัวแปรโดยการเขียนหรือวาดรูป

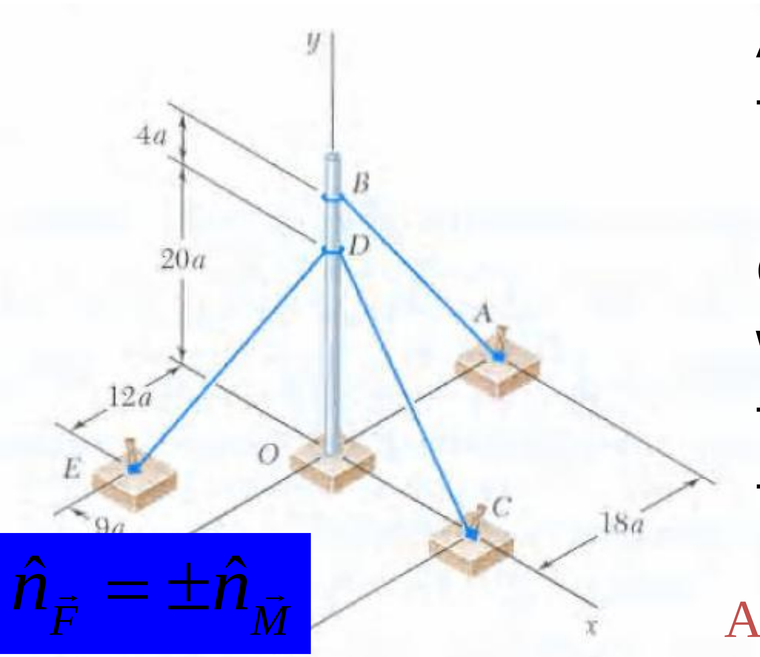
Reduction Summary

General force systems → Single force + single couple

2D force systems → Single force or single couple

simplest systems

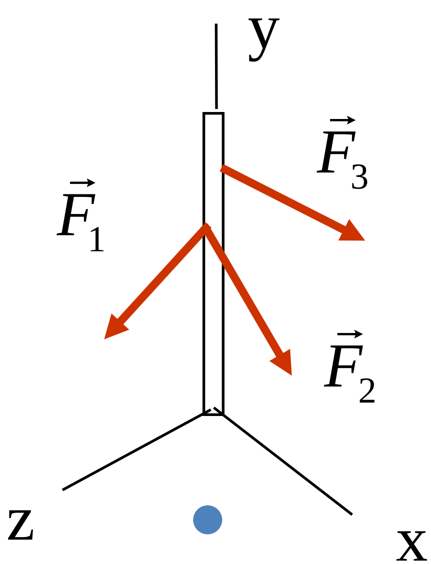
3D force systems → Wrench



$$\hat{n}_{\vec{F}} = \pm\hat{n}_{\vec{M}}$$

A flagpole is guyed by 3 cables. If the tensions in the cables have the same magnitude P (N), replace the forces exerted on the pole with an eqivalent wrench and determine the resultant force R and the point where the axis of the wrench intersects the x-z plane

Assume (x,0,z) is the point where wrench passes.



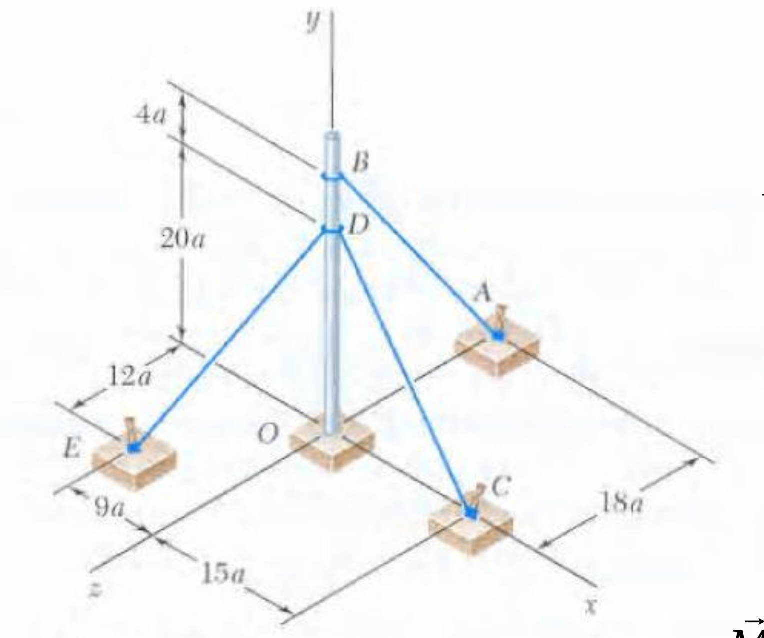
P: (x,0,z)

$$\vec{F}_1 = P\frac{a(-9\hat{i}+-20\hat{j}+12\hat{k})}{a\sqrt{9^2+20^2+12^2}} = \frac{P}{25}(-9\hat{i}+-20\hat{j}+12\hat{k})$$

$$\vec{F}_2 = P\frac{a(15\hat{i}-20\hat{j})}{a\sqrt{15^2+20^2}} = \frac{P}{25}(15\hat{i}-20\hat{j}) = \frac{P}{5}(3\hat{i}-4\hat{j})$$

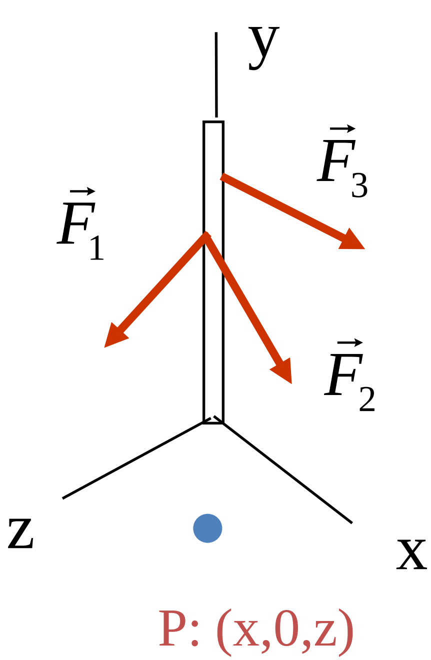
$$\vec{F}_2 = P\frac{a(-24\hat{j}-18\hat{k})}{a\sqrt{24^2+18^2}} = \frac{P}{30}(-24\hat{j}-18\hat{k}) = \frac{P}{5}(-4\hat{j}-3\hat{k})$$

$$\vec{R}_{Sys} = \vec{F}_1+\vec{F}_2+\vec{F}_3 = \frac{3P}{25}(2\hat{i}-20\hat{j}-\hat{k})$$

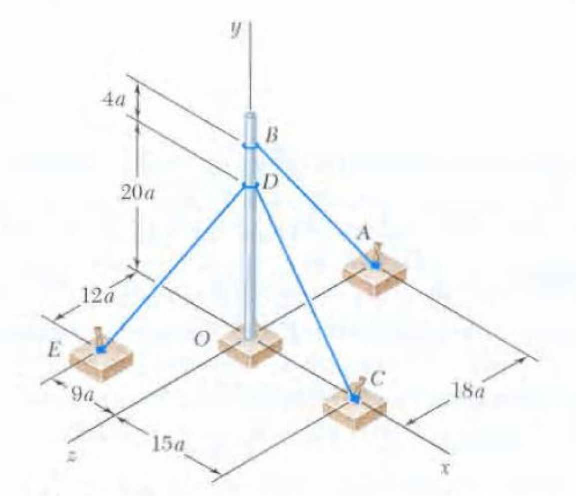


$$\vec{M}_P = (-x\hat{i} + 20a\hat{j} - z\hat{k}) \times \frac{P}{25}(-9\hat{i} + -20\hat{j} + 12\hat{k})$$
$$+ (-x\hat{i} + 20a\hat{j} - z\hat{k}) \times \frac{P}{5}(3\hat{i} - 4\hat{j})$$
$$+ (-x\hat{i} + 24a\hat{j} - z\hat{k}) \times \frac{P}{5}(-4\hat{j} - 3\hat{k})$$

$$\vec{M}_P = \frac{P}{25}\{(240a - 20z)\hat{i} + (9z + 12x)\hat{j} + (20x + 180a)\hat{k}\}$$
$$+ \frac{P}{5}\{(-4z)\hat{i} + (-3z)\hat{j} + (4x - 60a)\hat{k}\}$$
$$+ \frac{P}{5}\{(-72a - 4z)\hat{i} + (-3x)\hat{j} + (4x)\hat{k}\}$$



$$\vec{M}_P = \frac{P}{5}\{(-24a - 12z)\hat{i} + (\frac{-3x - 6z}{5})\hat{j} + (-24a + 12x)\hat{k}\}$$
$$= \frac{3P}{5}\{(-8a - 4z)\hat{i} + (\frac{-x - 2z}{5})\hat{j} + (-8a + 4x)\hat{k}\}$$



$$\vec{R}_{Sys} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{3P}{25}(2\hat{i} - 20\hat{j} - \hat{k})$$

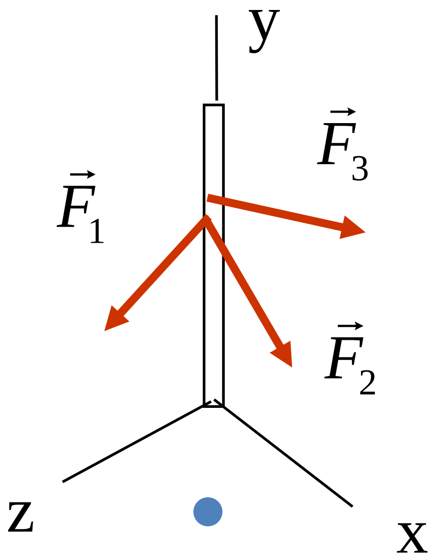
$$\vec{M}_P = \frac{3P}{5}\{(-8a - 4z)\hat{i} + (\frac{-x-2z}{5})\hat{j} + (-8a + 4x)\hat{k}\}$$

$$\boxed{\hat{n}_{\vec{F}} = \pm\hat{n}_{\vec{M}}}$$

$\boxed{\text{wrench condition}}$

$$\frac{1}{9\sqrt{5}}(2\hat{i} - 20\hat{j} - \hat{k}) = \pm\frac{1}{|\vec{M}_P|}\frac{3P}{5}\{(-8a - 4z)\hat{i} + (\frac{-x-2z}{5})\hat{j} + (-8a + 4x)\hat{k}\}$$

M (+ or – is ok)



P: (x,0,z)

$$\frac{1}{M}\frac{3P}{5}(-8a - 4z) = \frac{2}{9\sqrt{5}}$$

$$\frac{1}{M}\frac{3P}{5}\frac{(-x-2z)}{5} = -\frac{20}{9\sqrt{5}}$$

$$\frac{1}{M}\frac{3P}{5}(-8a + 4x) = -\frac{1}{9\sqrt{5}}$$

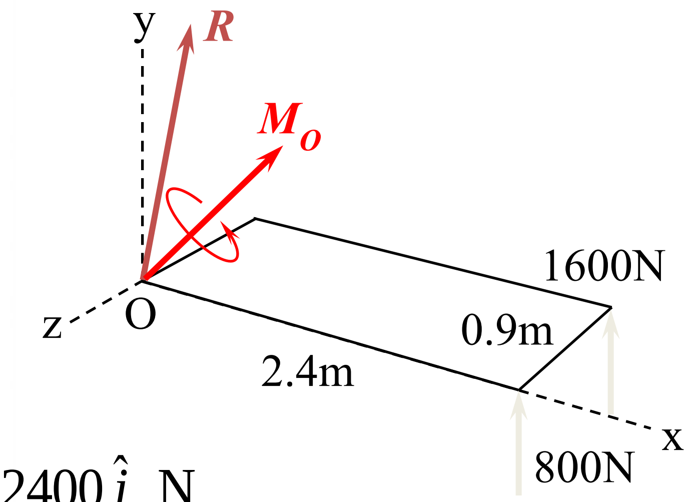
$$x = \frac{812}{405}a$$

$$z = -\frac{806}{405}a$$

$$M = 9\sqrt{5}\frac{3P}{5}(-\frac{8}{405}a) = -\frac{8\sqrt{5}}{75}Pa$$

141 Two upward loads are exerted on the small three-dimensional truss. Reduce these two loads to a single force–couple system at point O. Show that **R** is perpendicular to $\mathbf{M}_O$. Then determine the point in the x-z plane through which the resultant passes.

Ans. $\mathbf{R} = 2400\mathbf{j}$ N, $\mathbf{M}_O = 1440\mathbf{i} + 5760\mathbf{k}$ N·m
$x = 2.4$ m, $z = -0.6$ m



VD1

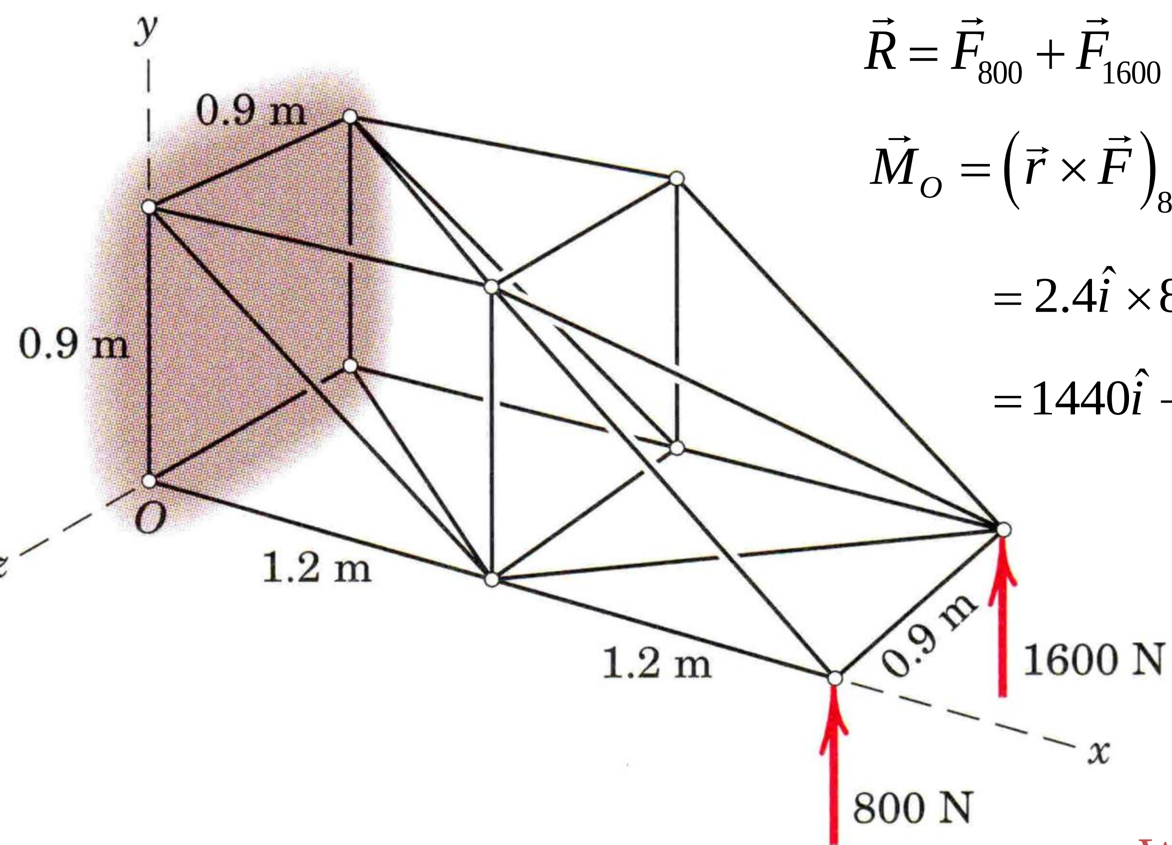
$$\vec{R} = \vec{F}_{800} + \vec{F}_{1600} = 2400\,\hat{j}\ \text{ N}$$

$$\vec{M}_O = \left(\vec{r}\times\vec{F}\right)_{800} + \left(\vec{r}\times\vec{F}\right)_{1600}$$

$$= 2.4\hat{i}\times 800\,\hat{j} + \left(2.4\hat{i} - 0.9\hat{k}\right)\times 1600\,\hat{j}$$

$$= 1440\hat{i} + 5760\hat{k}\quad \text{N-m}$$

$$\vec{R}\cdot\vec{M}_O = 0 \quad \Rightarrow \vec{R} \perp \vec{M}_O$$



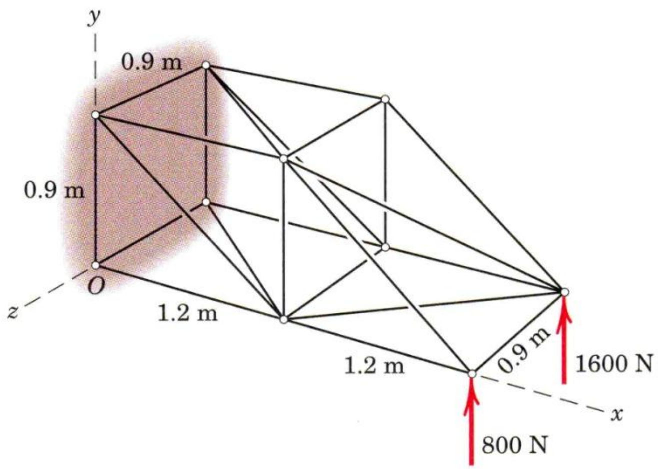
Problem 2/141

M has no component in the direction of ***R***.

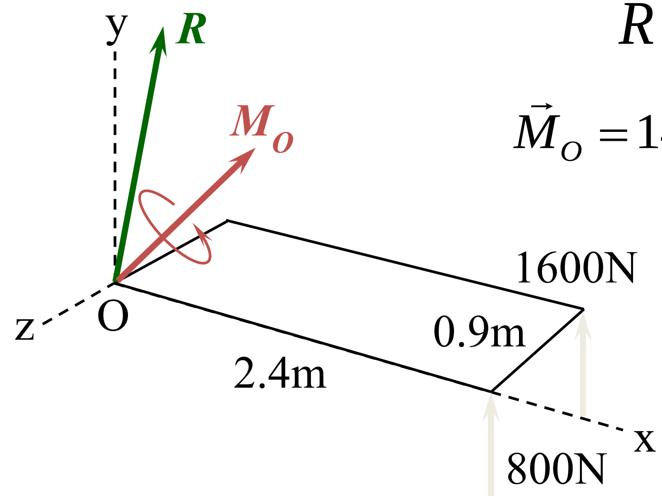
We can move ***R*** to new position to eliminate this couple completely

2/141 Two upward loads are exerted on the small three-dimensional truss. Reduce these two loads to a single force–couple system at point O. Show that $\mathbf{R}$ is perpendicular to $\mathbf{M}_O$. Then determine the point in the x-z plane through which the resultant passes.

Ans. $\mathbf{R} = 2400\mathbf{j}$ N, $\mathbf{M}_O = 1440\mathbf{i} + 5760\mathbf{k}$ N·m
$x = 2.4$ m, $z = -0.6$ m



Problem 2/141



VD1

$$\vec{R} = 2400\,\hat{j} \quad \text{N}$$

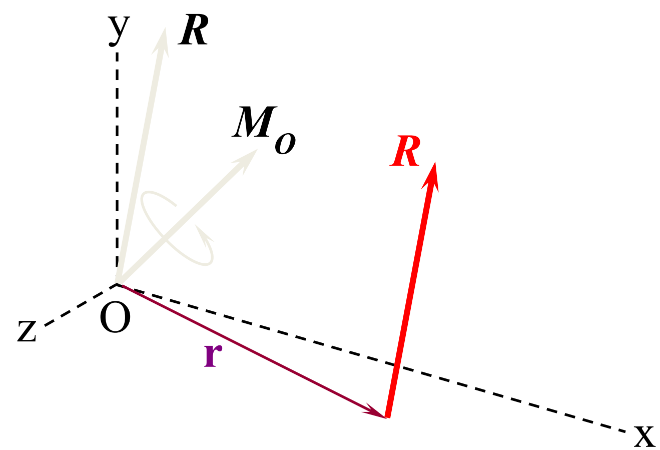
$$\vec{M}_O = 1440\hat{i} + 5760\hat{k} \quad \text{N-m}$$

$$\vec{M}_O + (-\vec{r}) \times \vec{R} = \vec{0} \qquad \Rightarrow \vec{M}_O = \vec{r} \times \vec{R}$$

$$1440\hat{i} + 5760\hat{k} = \left(r_x\hat{i} + r_y\hat{k}\right) \times \left(2400\,\hat{j}\right)$$

$$1440\hat{i} + 5760\hat{k} = 2400 r_x \hat{k} - 2400 r_y \hat{i}$$

$$r_x = 2.4 \qquad r_x = -0.6 \qquad \underline{\textbf{\textit{Ans}}}$$

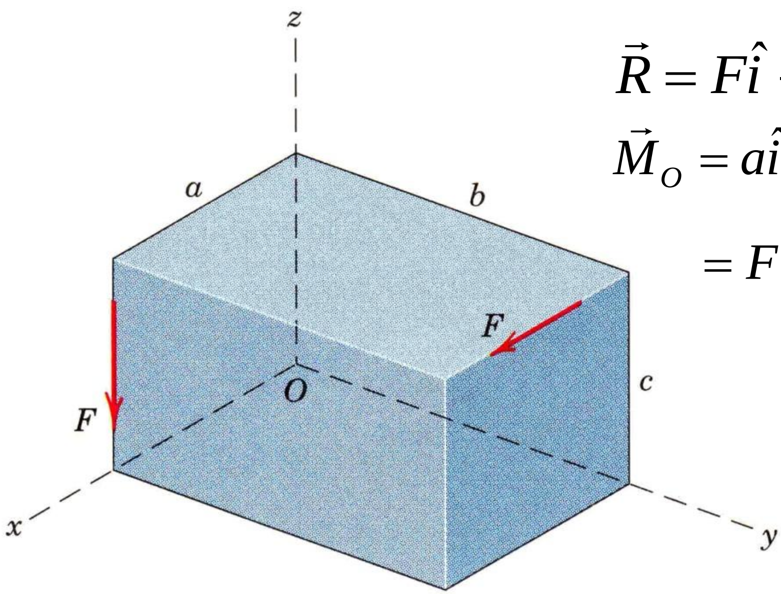


VD2

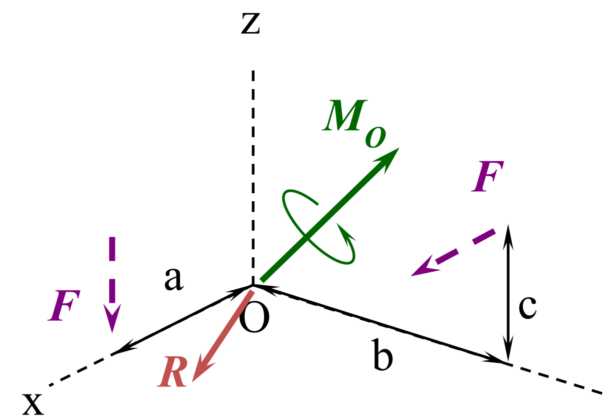
2/148 Replace the two forces acting on the rectangular solid by a wrench. Write the moment **M** associated with the wrench as a vector and specify the coordinates of the point P in the x-y plane through which the line of action of the wrench passes.

Ans. $\mathbf{M} = \frac{Fb}{2}(\mathbf{i} - \mathbf{k})$

$x = a + c,\ y = b/2$



Problem 2/148



VD1

$$\vec{R} = F\hat{i} - F\hat{k}$$

$$\vec{M}_O = a\hat{i} \times (-F)\hat{k} + \left(b\hat{j} + c\hat{k}\right) \times F\hat{i}$$

$$= F(a+c)\hat{j} - Fb\hat{k}$$

We move $\underline{\boldsymbol{R}}$ to the new location (x,y,z) where there is no couple.

$$\vec{M}_O + (\text{-}\vec{r}) \times \vec{R} = \vec{0} \quad \Rightarrow \vec{M}_O = \vec{r} \times \vec{R}$$

$$F(a+c)\hat{j} - Fb\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (F\hat{i} - F\hat{k})$$

$$F(a+c)\hat{j} - Fb\hat{k} = -yF\hat{i} + F(x+z)\hat{j} - Fy\hat{k}$$

$$\left.\begin{aligned} \hat{i} &: 0 = -yF \Rightarrow y = 0 \\ \hat{k} &: Fb = Fy \Rightarrow y = b \end{aligned}\right\}$$

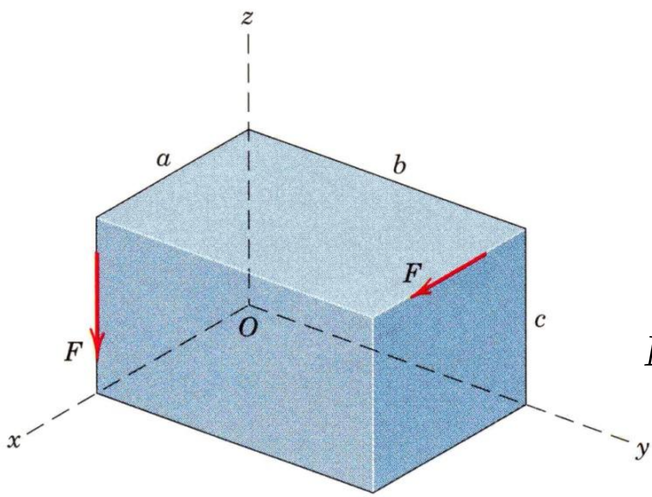
Generally b != 0, how come?

Generally in 3D, we can not change force-couple system to single-force system.

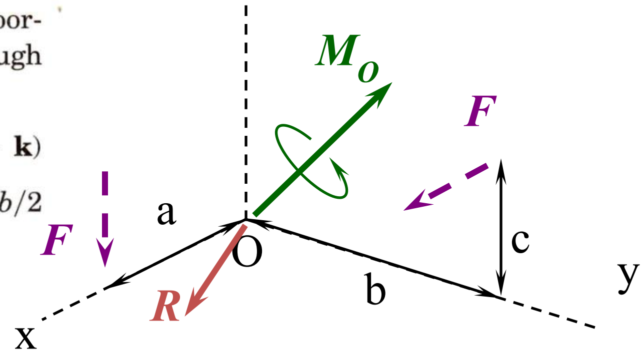
▶2/148 Replace the two forces acting on the rectangular solid by a wrench. Write the moment **M** associated with the wrench as a vector and specify the coordinates of the point P in the x-y plane through which the line of action of the wrench passes.

Ans. $\mathbf{M} = \dfrac{Fb}{2}(\mathbf{i} - \mathbf{k})$

$x = a + c,\ y = b/2$



Problem 2/148



VD1

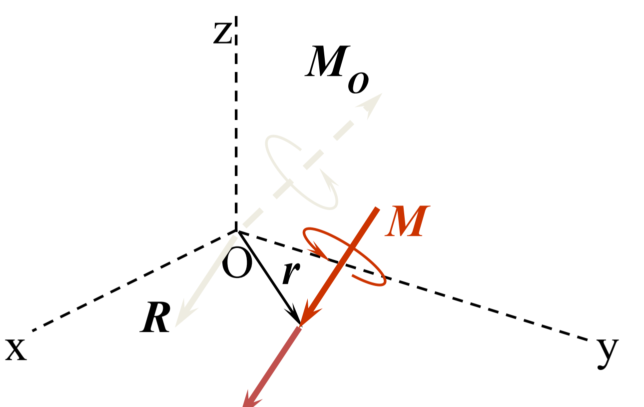
$$\vec{R} = F\hat{i} - F\hat{k}$$

$$\vec{M}_O = F(a+c)\hat{j} - Fb\hat{k}$$

$$\hat{n}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$$

$$\vec{M} = \left(\vec{M}_O \cdot \hat{n}_R\right)\hat{n}_R = \{(F(a+c)\hat{j} + Fb\hat{k}) \cdot \frac{1}{\sqrt{2}}(\hat{i} - \hat{k})\}\frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$$
$$= \frac{Fb}{2}(\hat{i} - \hat{k})$$

$$\vec{M}_{per} = \vec{M}_O - \vec{M} = -\left(\frac{Fb}{2}\right)\hat{i} + F(a+c)\hat{j} - \frac{Fb}{2}\hat{k}$$



VD2

$$\boxed{\vec{M}_{per} = \vec{r} \times \vec{R}}$$

$$-(Fb/2)\hat{i} + F(a+c)\hat{j} - (Fb/2)\hat{k} = \left(x_r\hat{i} + y_r\hat{j}\right) \times \left(F\hat{i} - F\hat{k}\right)$$

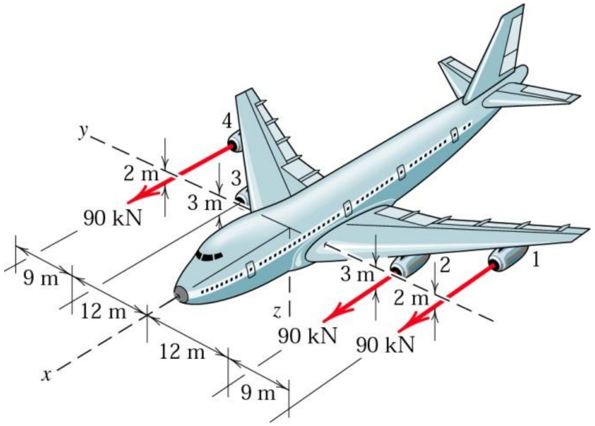
$$x_r = a + c \qquad y_r = b/2 \qquad \underline{\textbf{Ans}}$$

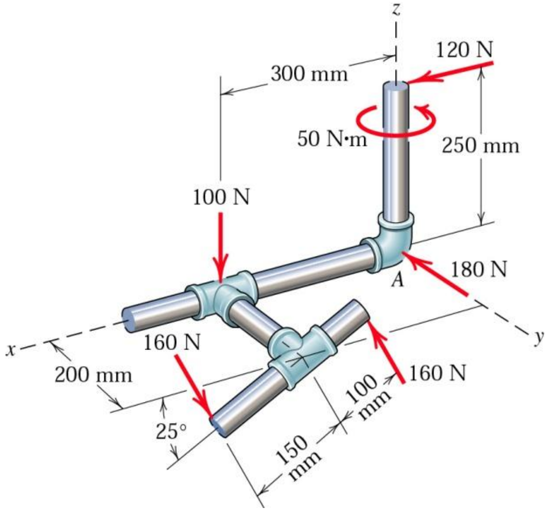
Note: we can calculate wrench just in 1 step see sample 2/16.

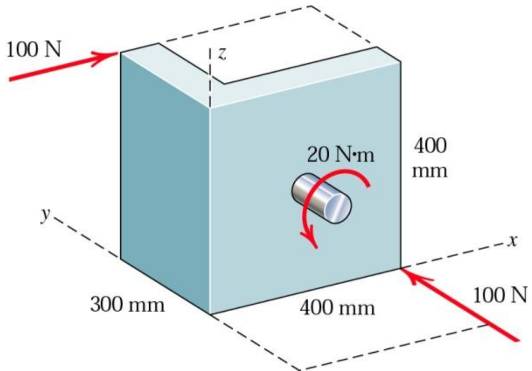
Recommended Problems

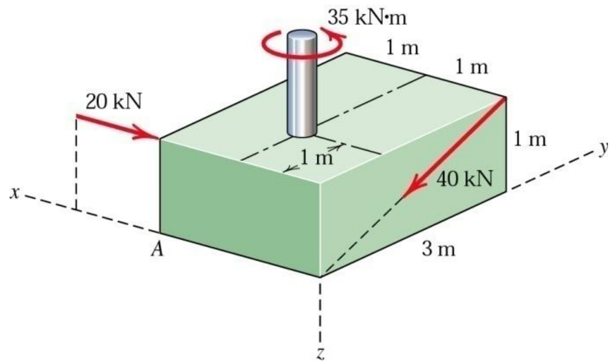
- 3D Resultants:

  2/140 2/142 2/149 2/150

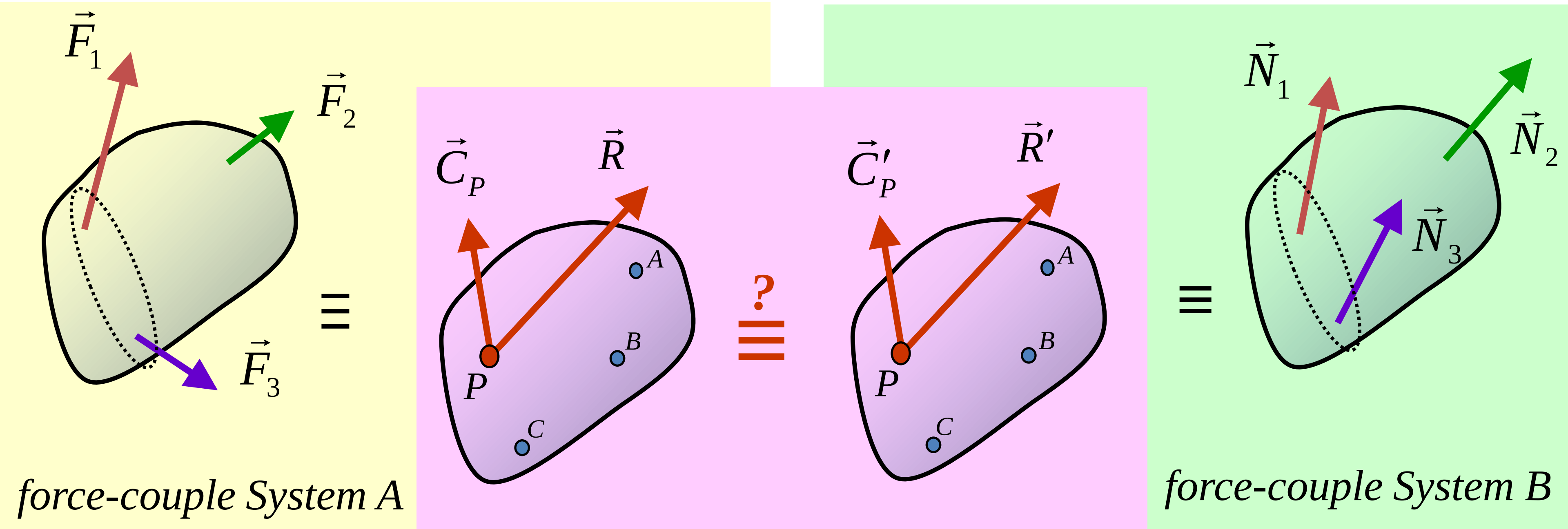








Equivalent System



In Statics Mechanics, we treat these two systems are equivalent if and only if

$$\boxed{\vec{R}_{Sys1} = \vec{R}_{Sys2}}$$

$$\boxed{\vec{M}_{Sys1} = \vec{M}_{Sys2} \text{ (for all point)}}$$

(in fact, just any one point is ok)

$$\boxed{\vec{R} = \vec{R}'} \quad \textit{(Pure Tendency to translate)}$$

$$\boxed{\vec{C}_P = \vec{C}'_P} \quad \textit{(Pure Tendency to rotate)}$$

$$(\textit{ i.e. } \vec{M}_P = \vec{M}'_P)$$

(just one point, and can be any point)

Equivalent System

useful for

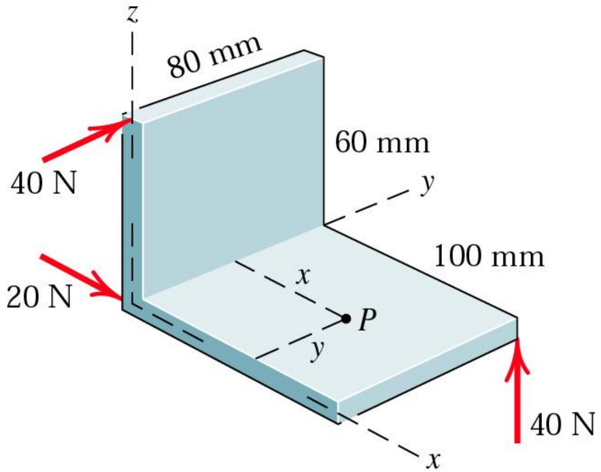
reducing any force-couple system

=> simplest resultant

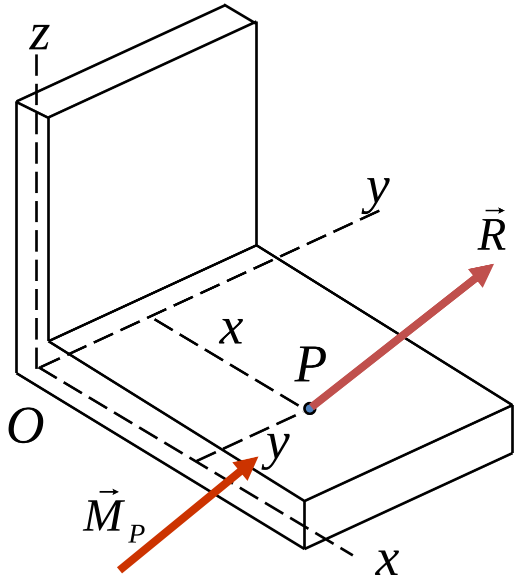
- General (3D) Force System
- Concurrent Force System
- Parallel Force System
- Coplanar Force System (2D System)

General-3D Force Systems

simplest system



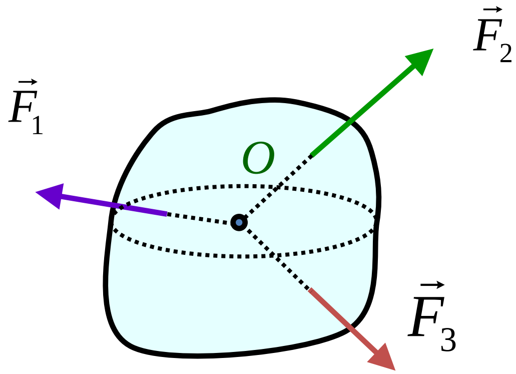
$\equiv$



$$\boxed{\vec{R}_{Sys1} = \vec{R}_{Sys2}}$$

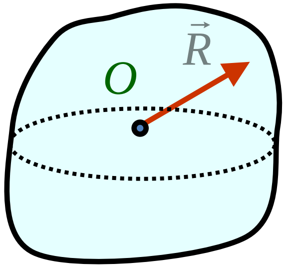
$$\vec{M}_{Sys1} = \vec{M}_{Sys2}$$
(for any one point)

Concurrent Force Systems (and no couple)



$\equiv$

simplest system



$$\boxed{\vec{R}_{Sys1} = \vec{R}_{Sys2}}$$

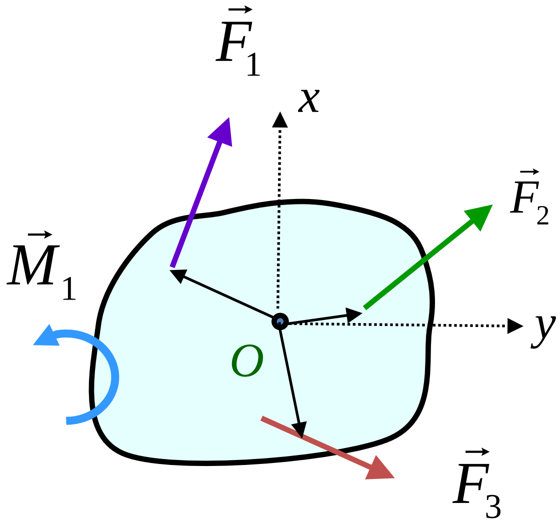
$$\vec{F}_R = \sum \vec{F}$$

$$\vec{M}_{Sys1} = \vec{M}_{Sys2}$$

(for any one point)

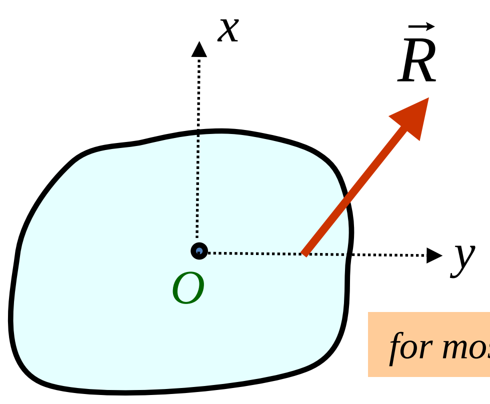
No benefit to use,
because it is satisfied by default
(moment at O)

Coplanner System



$\equiv$

simplest system



for most case (99.9%)

$$\boxed{\vec{R}_{Sys1} = \vec{R}_{Sys2}}$$

$\vec{M}_{Sys1} = \vec{M}_{Sys2}$

(for any one point)

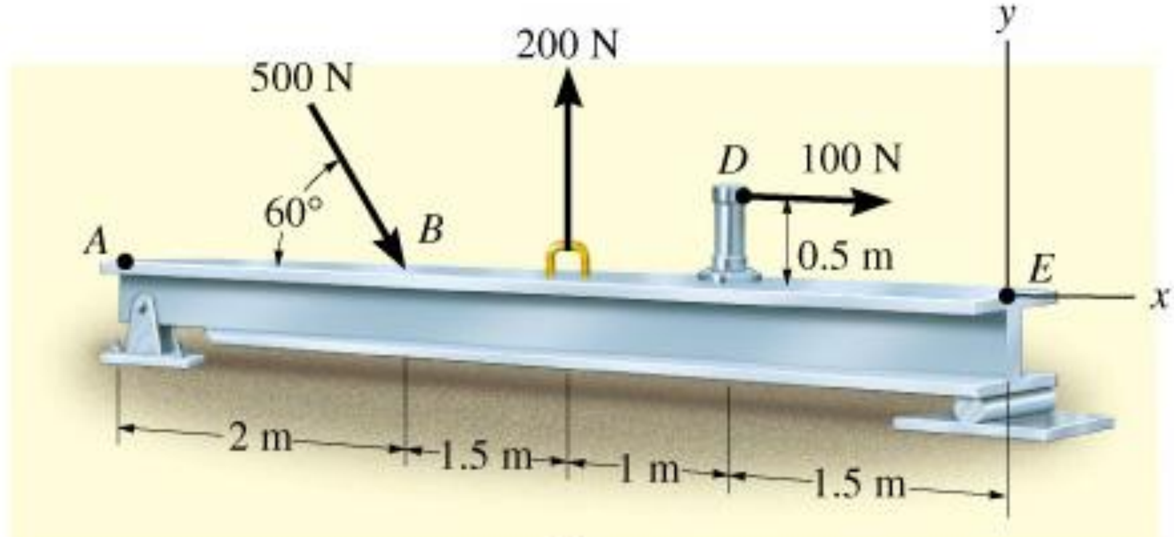
$$\vec{F}_R = \sum \vec{F}_i$$

$$\vec{M}_{R_O} = \sum \vec{M}_i + \sum (\vec{r}_i \times \vec{F}_i)$$

(Moment at point O)

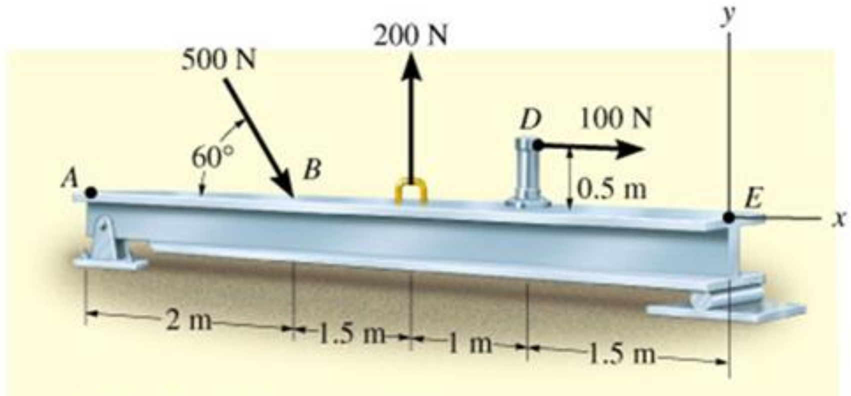
Example Hibbeler Ex 4-16 #1

Determine the magnitude, direction and location on the beam of a resultant force which is equivalent to the system of forces measured from *E*.

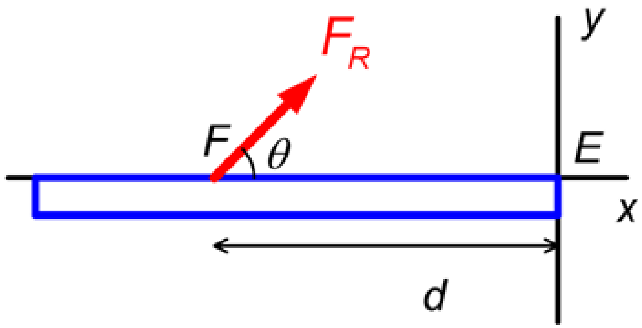


(a)

Example **Hibbeler Ex 4-16 #2**



System I



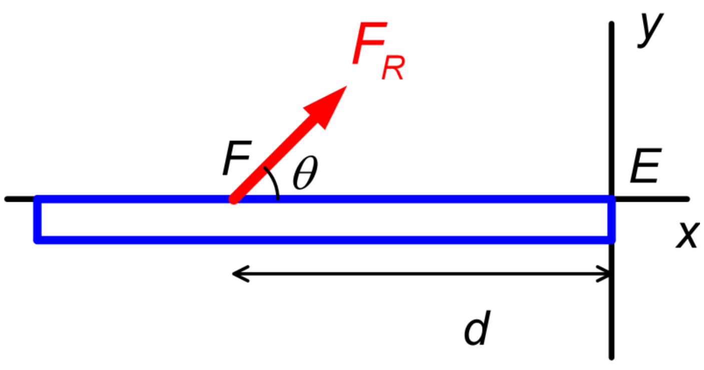
System II

$$\left[\sum F_{x,\text{sys II}} = \sum F_{x,\text{sys I}}\right] \quad F_R\cos\theta = (500\text{ N})\cos 60^\circ + (100\text{ N}) = 350\text{ N}$$

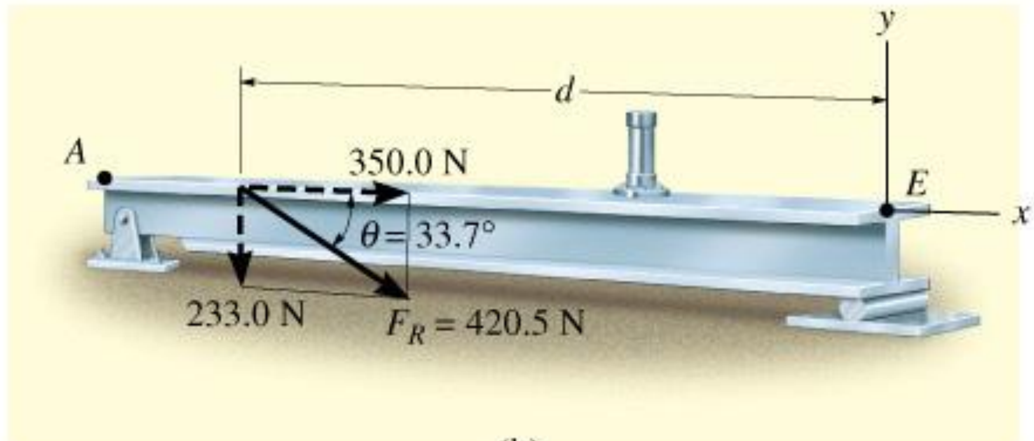
$$\left[\sum F_{y,\text{sys II}} = \sum F_{y,\text{sys I}}\right] \quad F_R\sin\theta = -(500\text{ N})\sin 60^\circ + (200\text{ N}) = -233.01\text{ N}$$

$$F_R = 420.47\text{ N} = 420\text{ N}$$

Example Hibbeler Ex 4-16 #3



System II



(b)

$$\tan\theta = \frac{F_R \sin\theta}{F_R \cos\theta} = \frac{-233.01\,\text{N}}{350\ \text{N}}$$

$$\theta = -33.653° = -33.7° = 33.7°\ \text{CW} \qquad \underline{\text{Ans}}$$

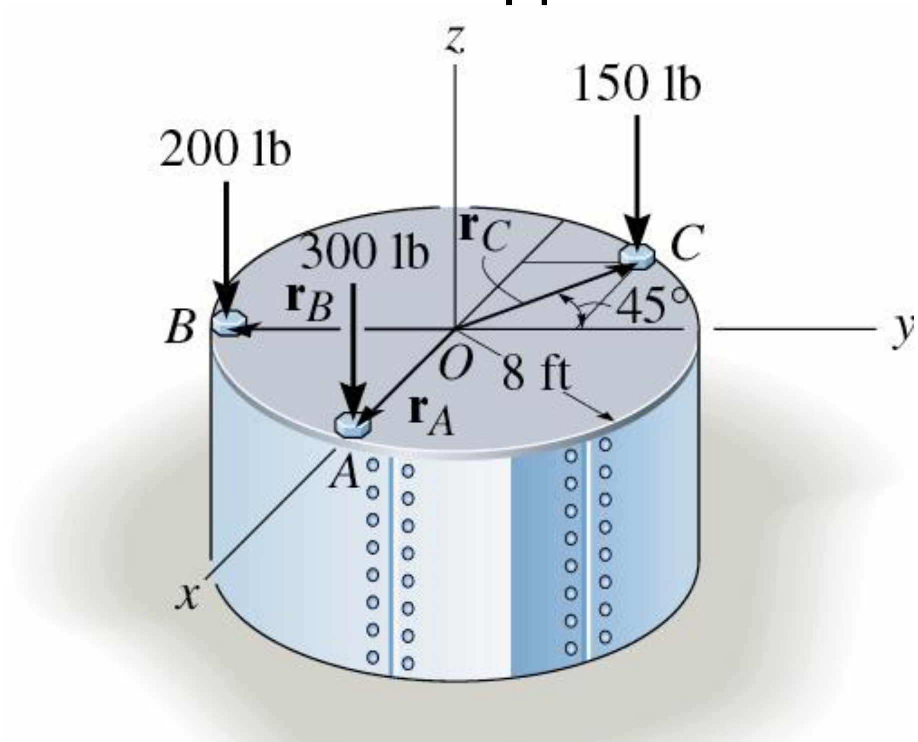
$$\left[\sum M_{E,\text{sys II}} = \sum M_{E,\text{sys I}}\right]$$

$$F_R \sin\theta(d) = (500\ \text{N})\sin 60°(4\ \text{m}) - (100\ \text{N})(0.5\ \text{m}) - (200\ \text{N})(2.5\ \text{m})$$

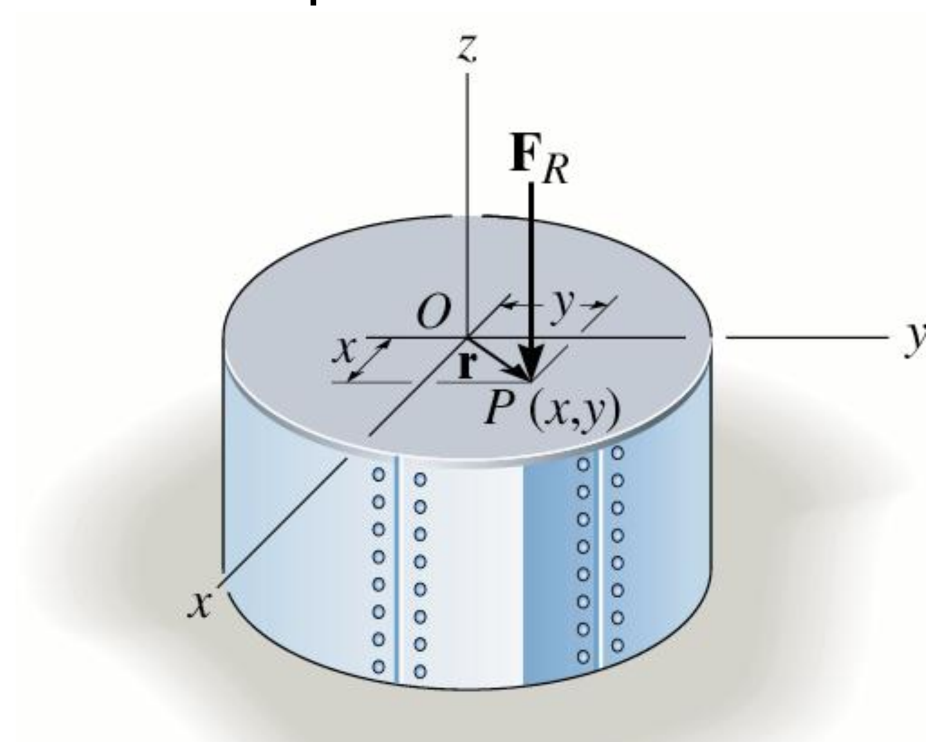
$$(233.01\,\text{N})(d) = 1182.25\ \text{N} \qquad \rightarrow\ d = 5.0730\ \text{m} = 5.07\ \text{m} \qquad \underline{\text{Ans}}$$

Example **Hibbeler Ex 4-19 #1**

Determine the magnitude and direction of a resultant equivalent to the given force system and locate its point of application *P* on the cover plate.



System I



System II

Example Hibbeler Ex 4-19 #2

$\vec{F}_A = -300\hat{k}$ lb, $\vec{r}_A = 8\hat{i}$ ft, $\vec{F}_B = -200\hat{k}$ lb, $\vec{r}_B = -8\hat{j}$ ft

$\vec{F}_C = -150\hat{k}$ lb, $\vec{r}_C = (-8\sin 45°\hat{i} + 8\cos 45°\hat{j})$ ft

$$\left[\sum \vec{F}_{\text{sys II}} = \sum \vec{F}_{\text{sys I}}\right]$$

$$\vec{F}_R = \vec{F}_A + \vec{F}_B + \vec{F}_C \quad \rightarrow \quad = -650\hat{k} \text{ lb} \qquad \underline{\text{Ans}}$$

$$\left[\sum \vec{M}_{O,\text{sys II}} = \sum \vec{M}_{O,\text{sys I}}\right]$$

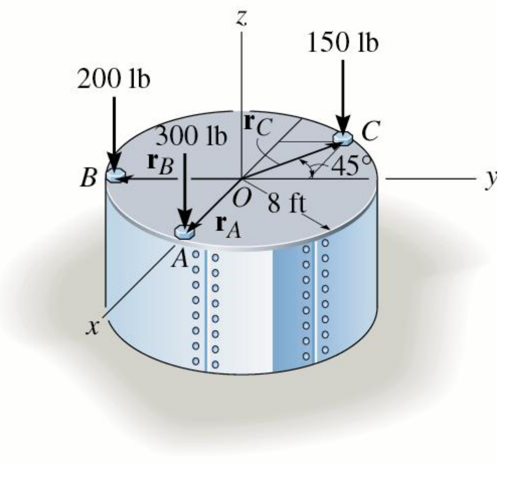
$$\vec{M}_{R_O} = \vec{r}_A \times \vec{F}_A + \vec{r}_B \times \vec{F}_B + \vec{r}_C \times \vec{F}_C$$

$$\vec{M}_{R_O} = (8\hat{i} \text{ ft}) \times (-300\hat{k} \text{ lb}) + (-8\hat{j} \text{ ft}) \times (-200\hat{k} \text{ lb}) +$$

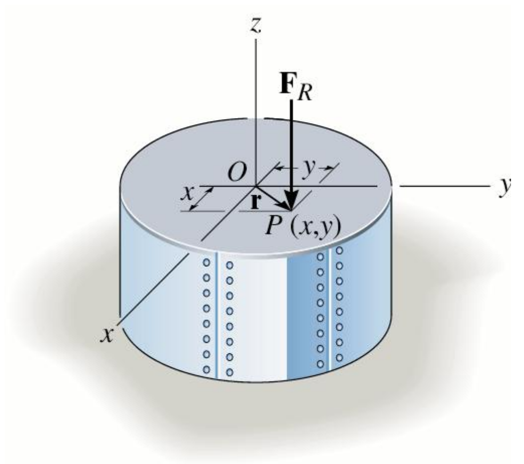
$$(-8\sin 45°\hat{i} + 8\cos 45°\hat{j}) \text{ ft} \times (-150\hat{k} \text{ lb})$$

$$= 2400\hat{j} + 1600\hat{i} - 848.53\hat{i} - 848.53\hat{j} \text{ lb} \cdot \text{ft}$$

$$= (751.47\hat{i} + 1551.5\hat{j}) \text{ lb} \cdot \text{ft} \qquad (1)$$



Example Hibbeler Ex 4-19 #3



$$\vec{M}_{R_O} = \vec{r} \times \vec{F}_R = (x\hat{i} + y\hat{j})\text{ ft} \times (-650\,\hat{k}\text{ lb})$$

$$= (-650y\hat{i} + 650x\hat{j})\text{ lb}\cdot\text{ft} \qquad (2)$$

From $(1) = (2)$

$$(-650y\hat{i} + 650x\hat{j})\text{ lb}\cdot\text{ft} = (751.47\hat{i} + 1551.5\,\hat{j})\text{ lb}\cdot\text{ft}$$

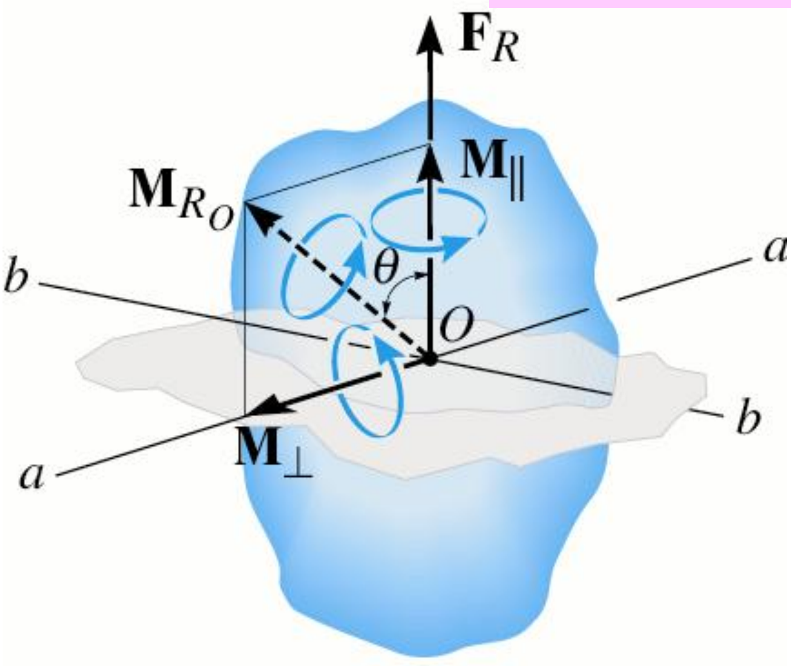
Equating $\hat{i}$ and $\hat{j}$ components

$$x = \frac{1551.5\text{ lb}\cdot\text{ft}}{650\text{ lb}} = 2.3869\text{ ft} = 2.39\text{ ft} \qquad \underline{\text{Ans}}$$

$$y = \frac{751.47\text{ lb}\cdot\text{ft}}{-650\text{ lb}} = -1.1561\text{ ft} = 1.16\text{ ft} \qquad \underline{\text{Ans}}$$

Reduction 3D System to a Wrench #1

$\vec{F}_R$ about O & $\vec{M}_{R_O}$ $\rightarrow$ $\vec{F}_R$ about O & $(\vec{M}_{\parallel} + \vec{M}_{\perp})$



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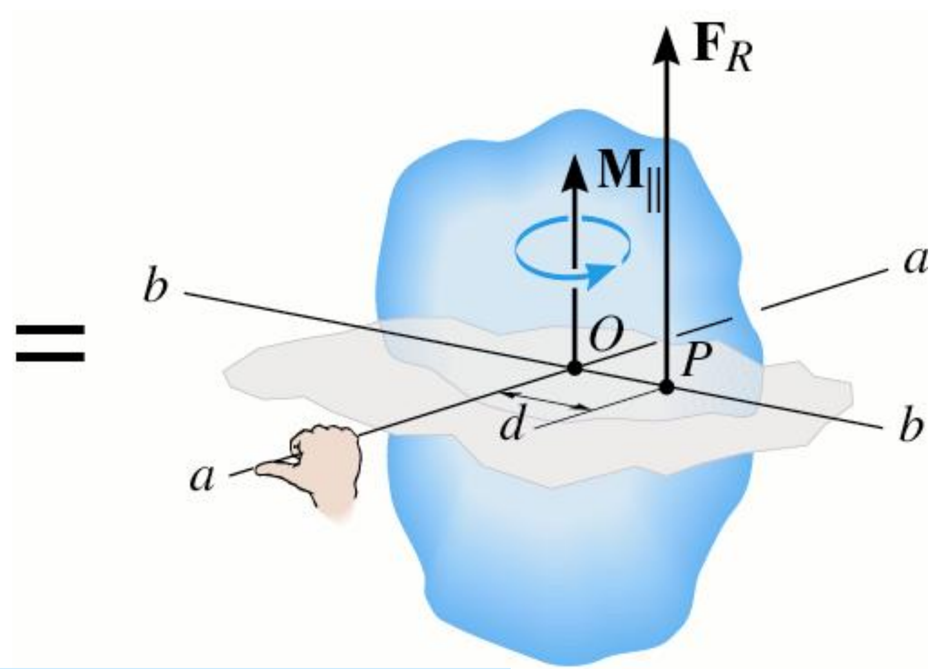
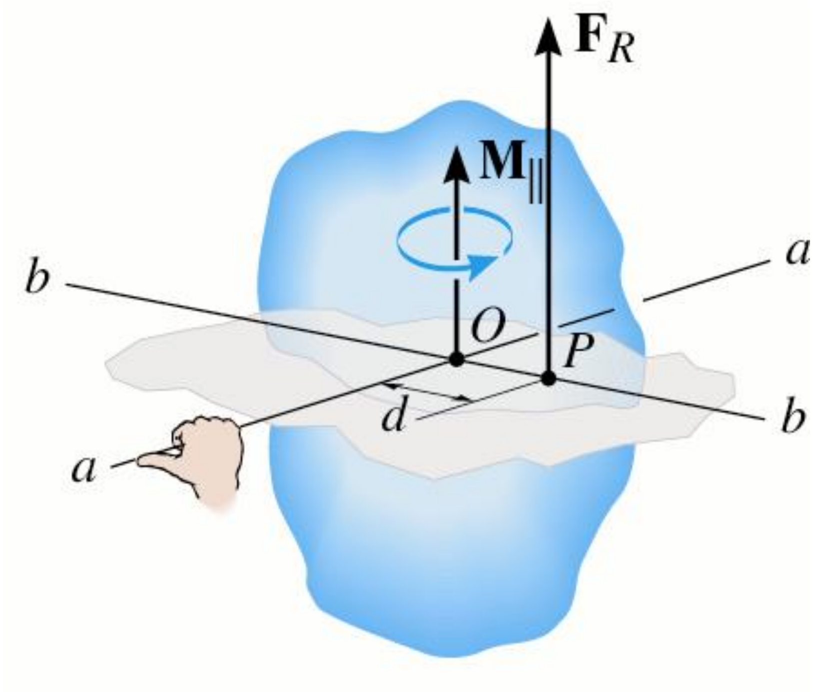


$\vec{F}_R$ about O & $(\vec{M}_{\parallel} + \vec{M}_{\perp})$ $\rightarrow$ $\vec{F}_R$ about P & $\vec{M}_{\parallel}$

Reduction 3D System to a Wrench #2



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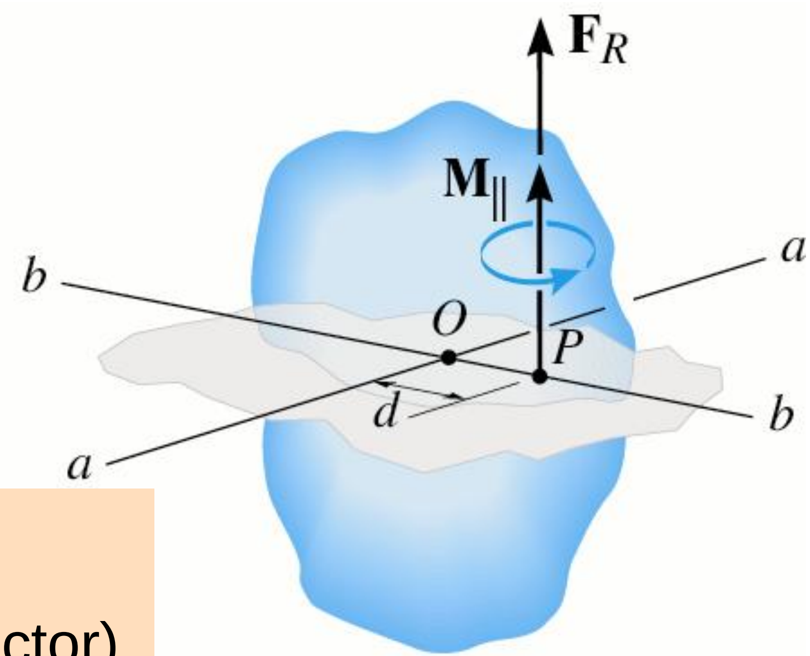

$\vec{F}_R$ about P & $\vec{M}_{||}$ $\rightarrow$

$\vec{F}_R$ about P & $\vec{M}_{||}$ about P (free vector)